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Invariant tensors for simple groups

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Abstract

The forms of the invariant primitive tensors for the simple Lie algebras A_l, B_l, C_l and D_l are investigated. A new family of symmetric invariant tensors is introduced using the non-trivial cocycles for the Lie algebra cohomology. For the A_l algebra it is explicitly shown that the generic forms of these tensors become zero except for the l primitive ones and that they give rise to the l primitive Casimir operators. Some recurrence and duality relations are given for the Lie algebra cocycles. Tables for the 3- and 5-cocycles for $su(3)$ and $su(4)$ are also provided. Finally, new relations involving the d and f $su(n)$ tensors are given.

1 Introduction

We devote this paper to a systematic study of the symmetric and skewsymmetric primitive invariant tensors that may be constructed on a compact simple Lie algebra $\mathcal{G}$. The symmetric invariant tensors give rise to the Casimirs of $\mathcal{G}$; the skewsymmetric ones determine the non-trivial cocycles for the Lie algebra cohomology (see *e.g.* [1]). It is well known [2, 3, 4, 5, 6, 7, 8, 9] that there are l such invariant symmetric primitive polynomials of order m_i ($i = 1, \ldots, l = \text{rank of } \mathcal{G}$), which determine l independent primitive Casimir operators of the same order, as well as l skewsymmetric invariant primitive tensors $\Omega^{(2m_i-1)}$ of order $(2m_i - 1)$. The latter determine the non-trivial cocycles for the Lie algebra cohomology, their order being related to the topological properties of the associated compact group manifold G which, from the point of view of the real homology, behaves as products of l $S^{(2m_i-1)}$ spheres [10, 11, 12, 13, 14, 15, 16, 17, 18]. The lowest examples of these tensors/polynomials ($m_1 = 2$) are the Killing tensor (which is a multiple of δ_{ij} for a compact algebra), the quadratic Casimir operator and the fully skewsymmetric structure constants of the simple algebra $\mathcal{G}$, which determine a three-cocycle on $\mathcal{G}$ (see Example 3.1 below).

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The study of the invariant primitive tensors on $\mathcal{G}$ (and especially in the most interesting case $\mathcal{G} = su(n)$) is not only mathematically relevant; their properties determine essential aspects of the physical theories based on the associated group G. These range from the form of the vertices in Feynman diagrams to the presence or absence of non-abelian anomalies in gauge theories (see in this last respect the articles in [19] and references therein; see also [20]). They are also relevant in other instances as *e.g.*, in (higher order) Yang-Mills duality problems [21], WZWN terms (see, *e.g.* [22] and references therein), $\mathcal{W}$-symmetry in conformal field theory [23], the construction of effective actions [24], etc. It is then convenient to have an explicit expression for the primitive tensors of the different orders as well as a convenient basis for the vector spaces of the invariant tensors of a given order.

The simplest way of obtaining an invariant symmetric tensor $k^{(r)}$ on $\mathcal{G}$ is by computing the trace of the symmetrised product of r generators. This symmetric trace (which may give a zero result depending on r and on the specific algebra $\mathcal{G}$ being considered) will not vanish, however, for arbitrary r. As a result, the trace will give rise to arbitrarily higher order tensors that cannot be primitive and independent of those of lower order m_i ($i = 1, \ldots, l$), and the same will apply to the Casimir operators constructed from them. This indicates that it is convenient to introduce a new family of tensors which is free of this problem. We shall do this in Sec. 3 by introducing a new family of symmetric invariant tensors $t^{(m_i)}$ from the l primitive $(2m_i - 1)$-cocycles $\Omega^{(2m_i-1)}$, $(i = 1, \ldots, l)$. These tensors will turn out to be 'orthogonal' in a precise sense (see Lemma 3.3). We shall also show how the l primitive Casimirs may be equivalently obtained from the $t^{(m_i)}$ tensors or from the cocycles $\Omega^{(2m_i-1)}$.

For the $su(n)$ algebras many results and techniques are already available. For instance, we can construct recursively [25] (see also [26, 4]) the so-called d-family of symmetric invariant tensors of order m starting from the symmetric d_{ijk} (for $su(n), n > 2$) and symmetrising the result $d_{(i_1 \ldots i_m)}$. This family (see Sec. 6.1) has a status like the k family *i.e.*, for n fixed and m large enough the $d^{(m)}$ tensors may be expressed as linear combinations of products of lower order ones. For fixed m and n (in $su(n)$) sufficiently large (in fact, for $n \geq m$) it is known how to define a basis of the vector spaces $\mathcal{V}^{(m)}$ of invariant symmetric tensors of order m, and via known identities for the $d^{(m)}$, how this basis reduces when $n < m$. We use these properties in our discussion of the $t^{(m_i)}$ family. In particular we see that a formal attempt to construct t-tensors of higher rank than is allowed from their definition necessarily yields an identically vanishing result (Sec.6.2). In proving this, we have needed to extend the set of identities for the d and f $su(n)$ tensors in the literature known to us.

The paper is organised as follows. After a short general discussion of the invariance properties of tensors and Casimir operators on $\mathcal{G}$ in Sec.2 the expression of the $(2m_i - 1)$-cocycles is given in Sec.3, where the $t^{(m_i)}$ family of invariant symmetric tensors is introduced. Sec.4 illustrates these general considerations for the $su(3)$ and $su(4)$ algebras. Sec.5 provides a general discussion of the primitivity of the invariant symmetric tensors and Casimir operators for the four infinite series A_l, B_l, C_l and D_l and shows explicitly, if not systematically, how a given primitive polynomial becomes algebraically dependent (non-primitive) when the rank of the algebra $\mathcal{G}$ is reduced sufficiently. Due to the special

relevance of the $su(n)$ algebras, Sec.6 is devoted to illustrate these ideas for the A_l case and, in particular, the usefulness of the $t^{(m_i)}$ family of tensors. Sec.7 discusses, again for the four infinite series, the properties of the $(2m_i-1)$-cocycles. The topological properties underlying the compact group manifolds provide a clue to establish, using the Hodge star $*$ operator, duality properties for the Lie algebra cocycles. This is done in Sec.8, and some of the formulae are illustrated by using the explicit results in Sec.4. Finally an Appendix develops some properties of the d and f $su(n)$ tensors for arbitrary n, and collects a number of new expressions which are needed for the derivation of crucial results in the main text.

2 Invariant symmetric polynomials and Casimir operators

Let $\mathcal{G}$ be a simple algebra of rank l with basis $\{X_i\}$, $[X_i, X_j] = C_{ij}^k X_k$, $i = 1, \ldots, r = \dim \mathcal{G}$, and let G be its (compact) associated Lie group[1]. Let $\{\omega^j\}$ be the dual basis in $\mathcal{G}^*$, $\omega^j(X_i) = \delta_i^j$, and consider a G-invariant symmetric tensor h of order m

$$h = h_{i_1 \ldots i_m} \omega^{i_1} \otimes \ldots \otimes \omega^{i_m} \quad . \tag{2.1}$$

The G-invariance of h means that

$$\sum_{s=1}^{m} C_{\nu i_s}^{\rho} h_{i_1 \ldots \widehat{i_s} \rho i_{s+1} \ldots i_m} = 0 \quad . \tag{2.2}$$

This is the case of the symmetric tensors $k^{(m)}$ given by the coordinates[2]

$$k_{i_1 \ldots i_m} = \frac{1}{m!} \mathrm{sTr}(X_{i_1} \ldots X_{i_m}) \equiv \mathrm{Tr}(X_{(i_1} \ldots X_{i_m)}) \quad , \tag{2.3}$$

where sTr is the symmetric trace, $\mathrm{sTr}(X_{i_1} \ldots X_{i_m}) = \sum_{\sigma \in S_m} \mathrm{Tr}(X_{i_{\sigma(1)}} \ldots X_{i_{\sigma(m)}})$, which is clearly ad-invariant[3]. In particular,

$$k_{ij} = \mathrm{Tr}(X_i X_j) = \kappa \delta_{ij} \tag{2.4}$$

[1] In the general discussions we adopt the 'mathematical' convention and take antihermitian generators X_i and hence negative definite Killing tensor $K_{ij} = \mathrm{Tr}(adX_i adX_j) \propto -\delta_{ij}$ since $\mathcal{G}$ is compact. When we consider explicit $su(n)$ examples, we follow the 'physical' convention and use hermitian generators $T_i, X_i = -iT_i$. In this case, an i accompanies the C_{ij}^k in the *r.h.s* of the commutators. When the generators are assumed to be in *matrix* form, we take them in the defining representation of the algebra. We always use unit metric and hence there is no distinction among upper and lower indices, their position being dictated by notational convenience.

[2] Indices inside round brackets $(i_1, \ldots, i_m)$ will always be understood as symmetrised with unit weight *i.e.*, with a factor $1/m!$. We use the same unit weight convention to antisymmetrise indices inside square brackets $[i_1, \ldots, i_m]$.

[3] We denote by $k^{(m)}$ the invariant symmetric tensors coming from the symmetric trace (2.3). In fact (see [5]) a complete set of l primitive (see below) invariant tensors may be constructed in this way by selecting suitable representations. Other families of symmetric invariant tensors will be identified by an appropriate letter, (*e.g.* t, d, v). Generic symmetric invariant polynomials are denoted by h.

where, for instance, $\kappa = -1/2$ for the generators X_i of the defining representation of $su(n)$.

Since $\mathcal{G} \sim T_e(G)$, the tangent space at the identity of G, we may use a left translation $L_g,\ g \in G$, to obtain a left-invariant (LI) m-tensor $h(g)$ on the group manifold G from the m-linear mapping $h : \mathcal{G} \times \overset{m}{\cdots} \times \mathcal{G} \to \mathbb{R}$. Its expression is the same as (2.1) where now the $\{\omega^j\}$ are replaced by the LI one-forms $\omega^i(g)$ on G, and (2.2) now follows from the fact that (see, *e.g.* [20])

$$L_{X_\nu(g)}\omega^\rho(g) = -C_{\nu i}^{\rho}\omega^i(g) \quad , \quad \nu, \rho, i = 1, \ldots, r \quad , \tag{2.5}$$

where $L_{X_\nu(g)}$ is the Lie derivative with respect to the vector field $X_\nu(g)$ obtained by applying the (tangent) left translation map L_g^T to $X_\nu(e) = X_\nu$; clearly the duality relation is maintained for the vector fields $\{X_i(g)\}$ and one-forms $\{\omega^j(g)\}$. In this context, the G-invariance condition (2.2) reads $L_{X_\nu(g)}h(g) = 0$.

Let G moreover be compact so that the Killing tensor may be taken as the unit matrix and let $h_{i_1 \ldots i_m}$ be an arbitrary symmetric invariant tensor. Then the order m element in the enveloping algebra $\mathcal{U}(\mathcal{G})$ defined by

$$\mathcal{C}^{(m)} = h^{i_1 \ldots i_m} X_{i_1} \ldots X_{i_m} \tag{2.6}$$

commutes with all elements in $\mathcal{G}$. This is so because the commutator $[X_\rho, \mathcal{C}^{(m)}]$ may be written as

$$[X_\rho, \mathcal{C}^{(m)}] = \sum_{s=1}^{m} C_{\rho\nu}^{i_s} h^{i_1 \ldots \widehat{i_s} \nu \ldots i_m} X_{i_1} \ldots X_{i_m} = 0 \quad , \tag{2.7}$$

which is indeed zero as a result of the invariance condition (2.2). In fact, the only conditions for the m-tensor h to generate a Casimir operator $\mathcal{C}^{(m)}$ of $\mathcal{G}$ of order m are its symmetry (non-symmetric indices would allow us to reduce the order m of $\mathcal{C}^{(m)}$ by replacing certain products of generators by commutators) and its invariance (eq. (2.7)); h does not need to be obtained from a symmetric trace (2.3). This leads to

Lemma 2.1 (*Casimirs and G-invariant symmetric polynomials*)
Let h be an invariant symmetric tensor of order m. Then, $\mathcal{C}^{(m)} = h^{i_1 \ldots i_m} X_{i_1} \ldots X_{i_m}$ is a Casimir of $\mathcal{G}$ of the same order m.

It is well known [2, 3, 6, 4, 5, 7, 8, 9] that a simple algebra of rank l has l independent (primitive) Casimir-Racah operators of order $m_1, \ldots, m_l$, the first of them given by the standard Casimir [27] operator $K_{ij}X^iX^j$ obtained from the Killing tensor ($m_1 = 2$) Thus, there must be (Cayley-Hamilton) relations among the invariant tensors obtained from (2.3) for $m > m_l$ or otherwise one would obtain an arbitrary number of primitive Casimirs. We shall study this problem in Sec.5 and apply our results to the $su(n)$ algebras in Sec.6.

3 Invariant skewsymmetric tensors and cocycles

Let $\theta(g) = \omega^i(g)X_i$ be the LI canonical form on a simple and compact group G, and consider the q-form $\mathrm{Tr}(\theta \wedge \overset{q}{\cdots} \wedge \theta)$. Due to the cyclic property of the trace and the anticommutativity

of one-forms, this form is zero for q even. Let q be odd. Then,

$$\Omega^{(q)}(g) = \frac{1}{q!}\mathrm{Tr}(\theta \wedge \overset{q}{\cdots} \wedge \theta) \tag{3.1}$$

is a closed form on the group manifold G, since $d\Omega \propto \mathrm{Tr}(\theta \wedge \overset{q+1}{\cdots} \wedge\theta) = 0$ on account of the Maurer-Cartan equations $d\theta = -\theta \wedge \theta$. Since $\Omega(g)$ is not exact (it cannot be the exterior differential of the $(q-1)$-form $\mathrm{Tr}(\theta \wedge \overset{q-1}{\cdots} \wedge\theta)$ which is zero because $q-1$ is even) it defines a Chevalley-Eilenberg [1] Lie algebra q-cocycle. If we set $q = 2m-1$, we find that

$$\begin{aligned}
\Omega^{(2m-1)}(g) &= \frac{1}{(2m-1)!}\mathrm{Tr}(X_{i_1}\dots X_{i_{2m-1}})\omega^{i_1}(g)\wedge\dots\wedge\omega^{i_{2m-1}}(g) \\
&= \frac{1}{(2m-1)!}\frac{1}{2^{m-1}}\mathrm{Tr}([X_{i_1},X_{i_2}][X_{i_3},X_{i_4}]\dots[X_{i_{2m-3}},X_{i_{2m-2}}]X_{i_{2m-1}}) \\
&\qquad \cdot\omega^{i_1}(g)\wedge\dots\wedge\omega^{i_{2m-1}}(g) \\
&= \frac{1}{(2m-1)!}\frac{1}{2^{m-1}}C_{i_1 i_2}^{l_1}\dots C_{i_{2m-3}i_{2m-2}}^{l_{m-1}}\mathrm{Tr}(X_{l_1}\dots X_{l_{m-1}}X_\sigma) \\
&\qquad \cdot\omega^{i_1}(g)\wedge\dots\wedge\omega^{i_{2m-2}}(g)\wedge\omega^\sigma(g) \quad .
\end{aligned} \tag{3.2}$$

The trace may in fact be replaced by a unit weight symmetric trace *i.e.*, by $\mathrm{Tr} \sim (1/m!)\mathrm{sTr}$ in (3.2), since the skewsymmetry in $i_1 \dots i_{2m-2}$ results in symmetry in $l_1 \dots l_{m-1}$. The factor $\frac{1}{2^{m-1}}$ is unimportant[4] and will be ignored from now on. Hence, we may define the $(2m-1)$-form on G representing a Lie algebra $(2m-1)$-cocycle by

$$\Omega^{(2m-1)}(g) = \frac{1}{(2m-1)!}\Omega_{i_1\dots i_{2m-2}\sigma}\omega^{i_1}(g)\wedge\dots\wedge\omega^{i_{2m-2}}(g)\wedge\omega^\sigma(g) \quad , \tag{3.3}$$

the coordinates of which are given by the constant skewsymmetric tensor

$$\Omega_{i_1\dots i_{2m-2}\sigma} = \frac{1}{(2m-1)!}\epsilon_{i_1\dots i_{2m-2}\sigma}^{j_1\dots j_{2m-2}\rho}C_{j_1 j_2}^{l_1}\dots C_{j_{2m-3}j_{2m-2}}^{l_{m-1}}k_{l_1\dots l_{m-1}\rho} \quad , \tag{3.4}$$

where $k_{l_1\dots l_{m-1}\rho}$ is the symmetric tensor of (2.3)[5] and the ϵ tensor is defined by

$$\epsilon_{\alpha_1\dots\alpha_n}^{\beta_1\dots\beta_n} = \sum_{\sigma\in S_n}(-1)^{\pi(\sigma)}\delta_{\alpha_1}^{\beta_{\sigma(1)}}\dots\delta_{\alpha_n}^{\beta_{\sigma(n)}} \quad , \tag{3.5}$$

where $\pi(\sigma)$ is the parity of the permutation σ. Although (3.4) is what follows naturally from (3.1) (setting aside the ignored factor $1/2^{(m-1)}$) it is convenient to notice that part

[4] We recall that the cohomology space is a vector space, and that numerical factors, although they relate *inequivalent* cocycles (different vectors in a given cohomology space $H^{(2m-1)}(\mathcal{G},\mathbb{R})$), are unimportant here; they determine only the normalisation of the different tensors.

[5] The invariance properties of (3.4) follow from the invariance of $k_{l_1\dots l_{m-1}\rho}$ and thus do not depend on the fact that it is expressed by a symmetric trace (2.3).

of the antisymmetrisation carried out by $\epsilon^{j_1\ldots j_{2m-2}\rho}_{i_1\ldots i_{2m-2}\sigma}$ is unnecessary, since (cf. (3.4)) may be rewritten as

$$\begin{aligned}\Omega_{\rho i_2\ldots i_{2m-2}\sigma} &= \frac{1}{(2m-3)!}\epsilon^{j_2\ldots j_{2m-2}}_{i_2\ldots i_{2m-2}} C^{l_1}_{\rho j_2}\ldots C^{l_{m-1}}_{j_{2m-3}j_{2m-2}} k_{l_1\ldots l_{m-1}\sigma} \\ &\equiv C^{l_1}_{\rho[i_2}\ldots C^{l_{m-1}}_{i_{2m-3}i_{2m-2}]} k_{l_1\ldots l_{m-1}\sigma}\end{aligned} \tag{3.6}$$

due to the skewsymmetry in ρ and σ of its *r.h.s.* This follows from the invariance of the symmetric polynomial $k_{l_1\ldots l_{m-1}\sigma}$ and is a generalisation of the simple $m = 2$ case for which (3.6) gives

$$\Omega_{\rho j\sigma} = k([X_\rho, X_j], X_\sigma) = k(X_\rho, [X_j, X_\sigma]) = -k([X_\sigma, X_j], X_\rho) = -\Omega_{\sigma j\rho} \quad . \tag{3.7}$$

Indeed, for an arbitrary invariant symmetric tensor h on $\mathcal{G}$ of order m we have from (3.6)

$$\begin{aligned}&(2m-3)!\,\Omega_{\rho i_2\ldots i_{2m-2}\sigma} = \epsilon^{j_2\ldots j_{2m-2}}_{i_2\ldots i_{2m-2}} h([X_\rho, X_{j_2}], [X_{j_3}, X_{j_4}], \ldots, [X_{j_{2m-3}}, X_{j_{2m-2}}], X_\sigma) \\ &= -\epsilon^{j_2\ldots j_{2m-2}}_{i_2\ldots i_{2m-2}} \sum_{s=2}^{m-1} h(X_\rho, [X_{j_3}, X_{j_4}], \ldots, [[X_{j_{2s-1}}, X_{j_{2s}}], X_{j_2}], \ldots, [X_{j_{2m-3}}, X_{j_{2m-2}}], X_\sigma) \\ &-\epsilon^{j_2\ldots j_{2m-2}}_{i_2\ldots i_{2m-2}} h(X_\rho, [X_{j_3}, X_{j_4}], \ldots, [X_{j_{2m-3}}, X_{j_{2m-2}}], [X_\sigma, X_{j_2}]) \\ &= \epsilon^{j_2\ldots j_{2m-2}}_{i_2\ldots i_{2m-2}} h(X_\rho, [X_{j_3}, X_{j_4}], \ldots, [X_{j_{2m-3}}, X_{j_{2m-2}}], [X_{j_2}, X_\sigma]) \\ &= -\epsilon^{j_2\ldots j_{2m-2}}_{i_2\ldots i_{2m-2}} h([X_\sigma, X_{j_2}], [X_{j_3}, X_{j_4}], \ldots, [X_{j_{2m-3}}, X_{j_{2m-2}}], X_\rho) = -(2m-3)!\,\Omega_{\sigma i_2\ldots i_{2m-2}\rho} \quad ,\end{aligned}$$

where we have used the invariance of h in the second equality, the Jacobi identity in the third (to see that every term in the summation symbol is zero) and the symmetry of h in the fourth one. The fact that the skewsymmetric tensors $\Omega^{(2m_i-1)}$ expressed by their coordinates (3.4) or (3.6) are indeed $(2m_i-1)$-cocycles follows from the Chevalley-Eilenberg approach to Lie algebra cohomology already mentioned [1]; for a direct proof which uses only the symmetry and invariance properties of the tensor used in its definition (3.6) see [28].

Example 3.1 Let $m = 2$. Using δ_{ij} (rather than k_{ij}) as the lowest order invariant polynomial, (3.6) gives

$$\Omega_{i_1 i_2\sigma} = C^{l_1}_{i_1 i_2}\delta_{l_1\sigma} = C_{i_1 i_2\sigma} \tag{3.8}$$

i.e., the three-cocycle $\Omega_{i_1 i_2 i_3}$ is determined by the structure constants of $\mathcal{G}$. It follows that the $3rd$ Lie algebra cohomology group $H^3(\mathcal{G})$ is non-zero for $\mathcal{G}$ simple, as is well known.

Example 3.2 Let $k_{i_1 i_2 i_3}$ be a $3rd$-order invariant symmetric polynomial. Such a $3rd$-order polynomial exists only for $su(n)$, $n > 2$ (this is the reason why in four dimensions only these groups are unsafe for non-abelian anomalies). Then, the $su(n)$ five-cocycle is given by (3.6)

$$\Omega_{\rho i_1 i_2 i_3\sigma} = \frac{1}{3!}\epsilon^{j_1 j_2 j_3}_{i_1 i_2 i_3} C^{l_1}_{\rho j_1} C^{l_2}_{j_2 j_3} k_{l_1 l_2\sigma} \quad . \tag{3.9}$$

The coordinates of $\Omega^{(3)}$, $\Omega^{(5)}$ for $su(3)$ and for $su(4)$ are given in tables 4.1, 4.3 and 4.4, 4.6 respectively. The expression of the three-cocycle follows directly from the structure constants, eq. (3.8); for five-cocycles we have used the symmetric Gell-Mann tensor d_{ijk} in (3.9) rather than k_{ijk} [$k_{ijk} = (1/3!)(-i/2)^3 \mathrm{sTr}(\lambda_i\lambda_j\lambda_k) = (i/4)d_{ijk}$, see (4.2) below].

Since the $(2m-1)$-forms $\Omega^{(2m-1)}(g)$ (3.3) are invariant, the coordinates of any $(2m-1)$-cocycle on the simple Lie algebra also satisfy the relation (cf. (2.2))

$$\sum_{s=1}^{2m-1} C^{\rho}_{\nu i_s}\Omega_{i_1\ldots\widehat{i_s}\rho i_{s+1}\ldots i_{2m-1}} = 0 \quad . \tag{3.10}$$

The simplest case corresponds to Example 3.1 for which

$$C^{\rho}_{\nu i_1}C_{\rho i_2 i_3} + C^{\rho}_{\nu i_2}C_{i_1\rho i_3} + C^{\rho}_{\nu i_3}C_{i_1 i_2\rho} = 0 \tag{3.11}$$

is the Jacobi identity $C^{\rho}_{\nu[i_1}C_{i_2 i_3]\rho} = 0$. Similarly, the five-cocycle (which only exists for $su(n)$, $n > 2$) satisfies

$$C^{\rho}_{\nu i_1}\Omega^{(5)}_{\rho i_2 i_3 i_4 i_5} + C^{\rho}_{\nu i_2}\Omega^{(5)}_{i_1\rho i_3 i_4 i_5} + C^{\rho}_{\nu i_3}\Omega^{(5)}_{i_1 i_2\rho i_4 i_5} + C^{\rho}_{\nu i_4}\Omega^{(5)}_{i_1 i_2 i_3\rho i_5} + C^{\rho}_{\nu i_5}\Omega^{(5)}_{i_1 i_2 i_3 i_4\rho} = 0 \quad . \tag{3.12}$$

Lemma 3.1
Let $h_{l_1\ldots l_m}$ be a symmetric G-invariant polynomial. Then,

$$\epsilon^{j_1\ldots j_{2m}}_{i_1\ldots i_{2m}} C^{l_1}_{j_1 j_2}\ldots C^{l_m}_{j_{2m-1}j_{2m}} h_{l_1\ldots l_m} = 0 \quad . \tag{3.13}$$

Proof: By replacing $C^{l_m}_{j_{2m-1}j_{2m}}h_{l_1\ldots lm}$ in the $l.h.s$ of (3.13) by the other terms in (2.2) we get

$$\epsilon^{j_1\ldots j_{2m}}_{i_1\ldots i_{2m}} C^{l_1}_{j_1 j_2}\ldots C^{l_{m-1}}_{j_{2m-3}j_{2m-2}}\Big(\sum_{s=1}^{m-1} C^{k}_{j_{2m-1}l_s} h_{l_1\ldots l_{s-1}k l_{s+1}\ldots l_{m-1}\,j_{2m}}\Big) \quad ,$$

which vanishes since all terms in the sum include products of the form $C^{s}_{jj'}C^{k}_{sj''}$ antisymmetrised in j, j', j'', which are zero due to the Jacobi identity, *q.e.d.* Note that if $h_{l_1\ldots l_m}$ is identified with $k_{l_1\ldots l_m}$ in (2.3) the equality follows immediately from the fact that *l.h.s.* (3.13) are the coordinates of $\Omega_{i_1\ldots i_{2m}}$, and $\Omega^{(2m)} \propto \mathrm{Tr}(\theta\wedge\overset{2m}{\cdots}\wedge\theta) = 0$.

Corollary 3.1
Let h now be a non-primitive symmetric G-invariant polynomial, *i.e.*, such that its coordinates are given by

$$h_{i_1\ldots i_p j_1\ldots j_q} = h^{(p)}_{(i_1\ldots i_p}h^{(q)}_{j_1\ldots j_q)} \quad , \tag{3.14}$$

where in the *r.h.s.* $(\ldots)$ indicates unit weight symmetrisation (see footnote 2) and $h^{(p)}$ and $h^{(q)}$ are symmetric invariant polynomials. Then the cocycle $\Omega^{2(p+q)-1}$ associated to (3.14) by (3.4) is zero.

Proof: $\Omega^{2(p+q)-1}$ is given by

$$\Omega^{2(p+q)-1}_{i_1\ldots i_{2(p+q)-1}} = \frac{1}{(2(p+q)-1)!}\epsilon^{j_1\ldots j_{2(p+q)-1}}_{i_1\ldots i_{2(p+q)-1}} C^{l_1}_{j_1 j_2}\ldots C^{l_p}_{j_{2p-1}j_{2p}} C^{m_1}_{j_{2p+1}j_{2p+2}}\ldots C^{m_{q-1}}_{j_{2(p+q)-3}j_{2(p+q)-2}}$$
$$h^{(p)}_{(l_1\ldots l_p} h^{(q)}_{m_1\ldots m_{q-1}j_{2(p+q)-1})}$$

and is zero by virtue of (3.13), *q.e.d.*

By definition, primitive tensors are not the product of lower order tensors, but may contain non-primitive terms. It follows from Corollary 3.1 that only the primitive term in k contributes to the cocycle (3.6). As a result, different families of symmetric tensors differing in non-primitive terms lead to proportional cocycles. Thus, and as far as the construction of the cocycles is concerned, we may use any family $h^{(m)}$ of symmetric invariant primitive tensors in (3.6), not necessarily that given by (2.3); for instance, for $su(n)$ we may use the d family of Sec 6.1. We shall use (3.6) with the definition (2.3) unless otherwise indicated.

We introduce now a new type of symmetric invariant polynomials by using the cocycles. Their interest will be made explicit in Sec. 6.

Lemma 3.2 (*Invariant symmetric polynomials from primitive cocycles*)
Let $\Omega^{(2m-1)}$ be a primitive cocycle. The l polynomials $t^{(m)}$ given by

$$t^{i_1\ldots i_m} = [\Omega^{(2m-1)}]^{j_1\ldots j_{2m-2}i_m} C^{i_1}_{j_1 j_2}\ldots C^{i_{m-1}}_{j_{2m-3}j_{2m-2}} \tag{3.15}$$

are G-invariant, symmetric and primitive. Moreover, they are traceless for $m>2$.

Proof: By construction $t^{i_1\ldots i_m}$ is an invariant polynomial (it is obtained by contracting the invariant tensors C and Ω). It follows from (3.15) that it is symmetric under interchange of the $(i_1\ldots i_{m-1})$ indices. Now

$$\begin{aligned}
t^{i_1\ldots i_m} &= [\Omega^{(2m-1)}]^{j_1\ldots j_{2m-2}i_m} C^{i_1}_{j_1 j_2} C^{i_2}_{j_3 j_4}\ldots C^{i_{m-1}}_{j_{2m-3}j_{2m-2}} \\
&= -\sum_{s=2}^{2m-2}([\Omega^{(2m-1)}]^{i_1 j_2\ldots \widehat{j_s}\rho\ldots j_{2m-2}i_m} C^{j_s}_{\rho j_2}) C^{i_2}_{j_3 j_4}\ldots C^{i_{m-1}}_{j_{2m-3}j_{2m-2}} \\
&\quad -[\Omega^{(2m-1)}]^{i_1 j_2\ldots j_{2m-2}\rho} C^{i_m}_{\rho j_2} C^{i_2}_{j_3 j_4}\ldots C^{i_{m-1}}_{j_{2m-3}j_{2m-2}} \\
&= -[\Omega^{(2m-1)}]^{i_1 j_2\ldots j_{2m-2}\rho} C^{i_m}_{\rho j_2} C^{i_2}_{j_3 j_4}\ldots C^{i_{m-1}}_{j_{2m-3}j_{2m-2}} \\
&= [\Omega^{(2m-1)}]^{\rho j_2\ldots j_{2m-2}i_1} C^{i_m}_{\rho j_2} C^{i_2}_{j_3 j_4}\ldots C^{i_{m-1}}_{j_{2m-3}j_{2m-2}} \\
&= t^{i_m i_2\ldots i_{m-1}i_1} \quad ,
\end{aligned}$$

where we have used the invariance (3.10) of $\Omega^{(2m-1)}$ in the second equality, the Jacobi identity in the third and the skewsymmetry of $\Omega^{(2m-1)}$ in the fourth. Thus t is invariant under the change $i_1 \leftrightarrow i_m$ and hence it is a completely symmetric tensor. For $m=2$ eq. (3.15) is proportional to the unit matrix since

$$C^{j_1 j_2 i_2} C^{i_1}_{j_1 j_2} = K^{i_1 i_2} \tag{3.16}$$

is the Killing tensor K. If we contract (3.15) with $\delta_{i_1 i_2}$ for $m > 2$ we see that

$$t_\sigma{}^{\sigma i_3 \ldots i_m} = 0 \tag{3.17}$$

by using the Jacobi identity for $C_{\sigma j_1 j_2} C^\sigma_{j_3 j_4}$, *q.e.d.*

Since $k_{i_1 i_2} \propto \delta_{i_1 i_2}$, the tracelessness of (3.15) may be seen as $t^{i_1 \ldots i_m}$ having zero contraction with k_{ij}. This extends to the full contractions with all higher order symmetric invariant tensors by means of the following

Lemma 3.3
Let $t^{i_1 \ldots i_m}$ be the symmetric invariant polynomial given by (3.15). Then

$$t^{i_1 \ldots i_l i_{l+1} \ldots i_m} t_{i_1 \ldots i_l} = 0 \quad , \quad \forall l < m \quad . \tag{3.18}$$

Proof: (3.18) is a consequence of Lemma 3.1, *q.e.d.*

Note. Eq. (3.18) implies the 'orthogonality' of the different polynomials $t^{(m)}_{i_1 \ldots i_m}$ obtained from $(2m-1)$-cocycles. Thus, a basis for the space $\mathcal{V}^{(m)}$ of invariant symmetric polynomials of order m is given by the symmetrised products of the primitive symmetric invariant polynomials (or their powers) leading to a symmetric invariant tensor of order m. We may express this as a

Corollary 3.2
The symmetric invariant primitive polynomial $t^{(m)}$ and the symmetrised products $t^{(m-r_1)} \otimes t^{(r_1)}$, $t^{(m-r_1-r_2)} \otimes t^{(r_1)} \otimes t^{(r_2)}$ (see (3.14)) etc., constitute a basis of the vector space $\mathcal{V}^{(m)}$ of the G-invariant symmetric polynomials on $\mathcal{G}$ of order m.

Example 3.3 For $su(n)$ the vector space of symmetric invariant polynomials of order six is given by $t^{(6)}_{i_1 \ldots i_6}$, $t^{(4)}_{(i_1 \ldots i_4}\delta_{i_5 i_6)}$, $t^{(3)}_{(i_1 i_2 i_3} t^{(3)}_{i_4 i_5 i_6)}$, $\delta_{(i_1 i_2}\delta_{i_4 i_5}\delta_{i_5 i_6)}$ ($t^{(3)}_{i_1 i_2 i_3}$ is proportional to $d_{i_1 i_2 i_3}$, see Sec.6.2).

As a consequence of Lemma 3.2, it follows that we can obtain Casimir operators $\mathcal{C}'$ from cocycles by means of

Corollary 3.3 (*Generalised Casimirs from primitive cocycles*)
The operator

$$\mathcal{C}'^{(m)} = [\Omega^{(2m-1)}]^{j_1 \ldots j_{2m-1}} X_{j_1} \ldots X_{j_{2m-1}} \tag{3.19}$$

is an m-order Casimir operator for $\mathcal{G}$.

Proof: Using the skewsymmetry of $\Omega^{(2m-1)}$ we rewrite (3.19) in the form

$$
\begin{aligned}
\mathcal{C}'^{(m)} &= [\Omega^{(2m-1)}]^{j_1\ldots j_{2m-1}} X_{j_1}\ldots X_{j_{2m-1}} \\
&= \frac{1}{2^{m-1}}[\Omega^{(2m-1)}]^{j_1\ldots j_{2m-2}\sigma}[X_{j_1},X_{j_2}]\ldots[X_{j_{2m-3}},X_{j_{2m-2}}]X_\sigma \\
&= \frac{1}{2^{m-1}}[\Omega^{(2m-1)}]^{j_1\ldots j_{2m-2}\sigma} C^{i_1}_{j_1 j_2}\ldots C^{i_{m-1}}_{j_{2m-3}j_{2m-2}} X_{i_1}\ldots X_{i_{m-1}}X_\sigma \\
&= \frac{1}{2^{m-1}} t^{i_1\ldots i_{m-1}\sigma} X_{i_1}\ldots X_{i_{m-1}}X_\sigma
\end{aligned}
\tag{3.20}
$$

and it follows that it is a Casimir as a consequence of Lemma 2.1, *q.e.d.*

4 The case of $su(n)$: coordinates of the $su(3)$- and $su(4)$-cocycles

Let us take as a basis $\{T_i\}$ of the $su(n)$ algebra the (n^2-1) traceless and hermitian $n\times n$ matrices of the defining representation of $su(n)$ satisfying the relations

$$
\begin{aligned}
&[T_i,T_j] = i f_{ij\cdot}{}^{k} T_k \quad , \quad \{T_i,T_j\} = c\delta_{ij} + d_{ij\cdot}{}^{k} T_k \quad , \\
&\mathrm{Tr}(T_iT_j) = \frac{1}{2}\delta_{ij} \quad , \quad T_iT_j = \frac{c}{2}\delta_{ij} + \frac{1}{2}d_{ij\cdot}{}^{k}T_k + \frac{i}{2}f_{ij\cdot}{}^{k}T_k \quad ,
\end{aligned}
\tag{4.1}
$$

where $i,j,k = 1,\ldots,n^2-1$ and $c \equiv \frac{1}{n}$ (for a study of the $su(3)$ tensors, see [29]). In the 'physical' (Gell-Mann) basis, it is customary to use the λ-matrices $\lambda_i = 2T_i$ for which the above relations trivially become

$$
\begin{aligned}
&[\lambda_i,\lambda_j] = 2i f_{ij\cdot}{}^{k}\lambda_k \quad , \quad \{\lambda_i,\lambda_j\} = 4c\delta_{ij} + 2d_{ij\cdot}{}^{k}\lambda_k \quad , \\
&\mathrm{Tr}(\lambda_i\lambda_j) = 2\delta_{ij} \quad , \quad \lambda_i\lambda_j = 2c\delta_{ij} + (d_{ij\cdot}{}^{k} + i f_{ij\cdot}{}^{k})\lambda_k \quad .
\end{aligned}
\tag{4.2}
$$

Using the Gell-Mann representation of $su(3)$ we have

Table 4.1: Non-zero structure constants for $su(3)$.

$f_{123}=1$	$f_{147}=1/2$	$f_{156}=-1/2$
$f_{246}=1/2$	$f_{257}=1/2$	$f_{345}=1/2$
$f_{367}=-1/2$	$f_{458}=\sqrt{3}/2$	$f_{678}=\sqrt{3}/2$

Table 4.2: $3rd$-order invariant symmetric polynomial for $su(3)$.

$d_{118} = 1/\sqrt{3}$	$d_{228} = 1/\sqrt{3}$	$d_{338} = 1/\sqrt{3}$	$d_{888} = -1/\sqrt{3}$
$d_{448} = -1/(2\sqrt{3})$	$d_{558} = -1/(2\sqrt{3})$	$d_{668} = -1/(2\sqrt{3})$	$d_{778} = -1/(2\sqrt{3})$
$d_{146} = 1/2$	$d_{157} = 1/2$	$d_{247} = -1/2$	$d_{256} = 1/2$
$d_{344} = 1/2$	$d_{355} = 1/2$	$d_{366} = -1/2$	$d_{377} = -1/2$

The f_{ijk} constitute the coordinates of the $su(3)$ three-cocycle. If we use the structure constants f_{ijk} and the symmetric d_{ijk} to define [*c.f.* (3.9)] the coordinates of the five-cocycle $\Omega^{(5)}$ by

$$\Omega^{(5)}_{i_1 i_2 i_3 i_4 i_5} = f^j{}_{i_1[i_2} f^k{}_{i_3 i_4]} d_{jki_5} \quad , \tag{4.3}$$

we obtain

Table 4.3: Non-zero coordinates of the $su(3)$ five-cocycle.

$\Omega_{12345} = 1/4$	$\Omega_{12367} = 1/4$	$\Omega_{12458} = \sqrt{3}/12,$
$\Omega_{12678} = -\sqrt{3}/12$	$\Omega_{13468} = -\sqrt{3}/12$	$\Omega_{13578} = -\sqrt{3}/12,$
$\Omega_{23478} = \sqrt{3}/12$	$\Omega_{23568} = -\sqrt{3}/12$	$\Omega_{45678} = -\sqrt{3}/6.$

Using the natural extension of the Gell-Mann labelling of generators to $su(4)$ in agreement with the contents of Table VI in [30] (where in its second part f should be replaced by d), we have

Table 4.4: Non-zero structure constants for $su(4)$.

$f_{1,2,3} = 1$	$f_{1,4,7} = 1/2$	$f_{1,5,6} = -1/2$
$f_{1,9,12} = 1/2$	$f_{1,10,11} = -1/2$	$f_{2,4,6} = 1/2$
$f_{2,5,7} = 1/2$	$f_{2,9,11} = 1/2$	$f_{2,10,12} = 1/2$
$f_{3,4,5} = 1/2$	$f_{3,6,7} = -1/2$	$f_{3,9,10} = 1/2$
$f_{3,11,12} = -1/2$	$f_{4,5,8} = \sqrt{3}/2$	$f_{4,9,14} = 1/2$
$f_{4,10,13} = -1/2$	$f_{5,9,13} = 1/2$	$f_{5,10,14} = 1/2$
$f_{6,7,8} = \sqrt{3}/2$	$f_{6,11,14} = 1/2$	$f_{6,12,13} = -1/2$
$f_{7,11,13} = 1/2$	$f_{7,12,14} = 1/2$	$f_{8,9,10} = 1/(2\sqrt{3})$
$f_{8,11,12} = 1/(2\sqrt{3})$	$f_{8,13,14} = -1/\sqrt{3}$	$f_{9,10,15} = \sqrt{2}/\sqrt{3}$
$f_{11,12,15} = \sqrt{2}/\sqrt{3}$	$f_{13,14,15} = \sqrt{2}/\sqrt{3}$	

Table 4.5: $3rd$-order invariant symmetric polynomial for $su(4)$.

$d_{4,4,3} = 1/2$	$d_{5,5,3} = 1/2$	$d_{6,6,3} = -1/2$	$d_{7,7,3} = -1/2$
$d_{9,9,3} = 1/2$	$d_{10,10,3} = 1/2$	$d_{11,11,3} = -1/2$	$d_{12,12,3} = -1/2$
$d_{1,1,8} = 1/\sqrt{3}$	$d_{2,2,8} = 1/\sqrt{3}$	$d_{3,3,8} = 1/\sqrt{3}$	$d_{4,4,8} = -1/(2\sqrt{3})$
$d_{5,5,8} = -1/(2\sqrt{3})$	$d_{6,6,8} = -1/(2\sqrt{3})$	$d_{7,7,8} = -1/(2\sqrt{3})$	$d_{8,8,8} = -1/\sqrt{3}$
$d_{9,9,8} = 1/(2\sqrt{3})$	$d_{10,10,8} = 1/(2\sqrt{3})$	$d_{11,11,8} = 1/(2\sqrt{3})$	$d_{12,12,8} = 1/(2\sqrt{3})$
$d_{13,13,8} = -1/\sqrt{3}$	$d_{14,14,8} = -1/\sqrt{3}$	$d_{1,1,15} = 1/\sqrt{6}$	$d_{2,2,15} = 1/\sqrt{6}$
$d_{3,3,15} = 1/\sqrt{6}$	$d_{4,4,15} = 1/\sqrt{6}$	$d_{5,5,15} = 1/\sqrt{6}$	$d_{6,6,15} = 1/\sqrt{6}$
$d_{7,7,15} = 1/\sqrt{6}$	$d_{8,8,15} = 1/\sqrt{6}$	$d_{9,9,15} = -1/\sqrt{6}$	$d_{10,10,15} = -1/\sqrt{6}$
$d_{11,11,15} = -1/\sqrt{6}$	$d_{12,12,15} = -1/\sqrt{6}$	$d_{13,13,15} = -1/\sqrt{6}$	$d_{14,14,15} = -1/\sqrt{6}$
$d_{15,15,15} = -2/\sqrt{6}$	$d_{1,4,6} = 1/2$	$d_{1,5,7} = 1/2$	$d_{1,9,11} = 1/2$
$d_{1,10,12} = 1/2$	$d_{2,4,7} = -1/2$	$d_{2,5,6} = 1/2$	$d_{2,9,12} = -1/2$
$d_{2,10,11} = 1/2$	$d_{4,9,13} = 1/2$	$d_{4,10,14} = 1/2$	$d_{5,9,14} = -1/2$
$d_{5,10,13} = 1/2$	$d_{6,11,13} = 1/2$	$d_{6,12,14} = 1/2$	$d_{7,11,14} = -1/2$
$d_{7,12,13} = 1/2$			

Table 4.6: Non-zero coordinates of the $su(4)$ five-cocycle.

$\Omega_{1,2,3,4,5} = 1/4$	$\Omega_{1,2,3,6,7} = 1/4$	$\Omega_{1,2,3,9,10} = 1/4$
$\Omega_{1,2,3,11,12} = 1/4$	$\Omega_{1,2,4,5,8} = \sqrt{3}/12$	$\Omega_{1,2,4,9,14} = 1/12$
$\Omega_{1,2,4,10,13} = -1/12$	$\Omega_{1,2,5,9,13} = 1/12$	$\Omega_{1,2,5,10,14} = 1/12$
$\Omega_{1,2,6,7,8} = -\sqrt{3}/12$	$\Omega_{1,2,6,11,14} = -1/12$	$\Omega_{1,2,6,12,13} = 1/12$
$\Omega_{1,2,7,11,13} = -1/12$	$\Omega_{1,2,7,12,14} = -1/12$	$\Omega_{1,2,8,9,10} = \sqrt{3}/36$
$\Omega_{1,2,8,11,12} = -\sqrt{3}/36$	$\Omega_{1,2,9,10,15} = \sqrt{6}/18$	$\Omega_{1,2,11,12,15} = -\sqrt{6}/18$
$\Omega_{1,3,4,6,8} = -\sqrt{3}/12$	$\Omega_{1,3,4,11,13} = 1/12$	$\Omega_{1,3,4,12,14} = 1/12$
$\Omega_{1,3,5,7,8} = -\sqrt{3}/12$	$\Omega_{1,3,5,11,14} = -1/12$	$\Omega_{1,3,5,12,13} = 1/12$
$\Omega_{1,3,6,9,13} = -1/12$	$\Omega_{1,3,6,10,14} = -1/12$	$\Omega_{1,3,7,9,14} = 1/12$
$\Omega_{1,3,7,10,13} = -1/12$	$\Omega_{1,3,8,9,11} = -\sqrt{3}/36$	$\Omega_{1,3,8,10,12} = -\sqrt{3}/36$

$\Omega_{1,3,9,11,15} = -\sqrt{6}/18$	$\Omega_{1,3,10,12,15} = -\sqrt{6}/18$	$\Omega_{1,4,5,9,12} = 1/12$
$\Omega_{1,4,5,10,11} = -1/12$	$\Omega_{1,4,7,9,10} = 1/12$	$\Omega_{1,4,7,11,12} = 1/12$
$\Omega_{1,4,7,13,14} = 1/12$	$\Omega_{1,4,8,11,13} = -\sqrt{3}/36$	$\Omega_{1,4,8,12,14} = -\sqrt{3}/36$
$\Omega_{1,4,11,13,15} = \sqrt{6}/36$	$\Omega_{1,4,12,14,15} = \sqrt{6}/36$	$\Omega_{1,5,6,9,10} = -1/12$
$\Omega_{1,5,6,11,12} = -1/12$	$\Omega_{1,5,6,13,14} = -1/12$	$\Omega_{1,5,8,11,14} = \sqrt{3}/36$
$\Omega_{1,5,8,12,13} = -\sqrt{3}/36$	$\Omega_{1,5,11,14,15} = -\sqrt{6}/36$	$\Omega_{1,5,12,13,15} = \sqrt{6}/36$
$\Omega_{1,6,7,9,12} = 1/12$	$\Omega_{1,6,7,10,11} = -1/12$	$\Omega_{1,6,8,9,13} = -\sqrt{3}/36$
$\Omega_{1,6,8,10,14} = -\sqrt{3}/36$	$\Omega_{1,6,9,13,15} = \sqrt{6}/36$	$\Omega_{1,6,10,14,15} = \sqrt{6}/36$
$\Omega_{1,7,8,9,14} = \sqrt{3}/36$	$\Omega_{1,7,8,10,13} = -\sqrt{3}/36$	$\Omega_{1,7,9,14,15} = -\sqrt{6}/36$
$\Omega_{1,7,10,13,15} = \sqrt{6}/36$	$\Omega_{1,9,12,13,14} = -1/12$	$\Omega_{1,10,11,13,14} = 1/12$
$\Omega_{2,3,4,7,8} = \sqrt{3}/12$	$\Omega_{2,3,4,11,14} = 1/12$	$\Omega_{2,3,4,12,13} = -1/12$
$\Omega_{2,3,5,6,8} = -\sqrt{3}/12$	$\Omega_{2,3,5,11,13} = 1/12$	$\Omega_{2,3,5,12,14} = 1/12$
$\Omega_{2,3,6,9,14} = 1/12$	$\Omega_{2,3,6,10,13} = -1/12$	$\Omega_{2,3,7,9,13} = 1/12$
$\Omega_{2,3,7,10,14} = 1/12$	$\Omega_{2,3,8,9,12} = \sqrt{3}/36$	$\Omega_{2,3,8,10,11} = -\sqrt{3}/36$
$\Omega_{2,3,9,12,15} = \sqrt{6}/18$	$\Omega_{2,3,10,11,15} = -\sqrt{6}/18$	$\Omega_{2,4,5,9,11} = 1/12$
$\Omega_{2,4,5,10,12} = 1/12$	$\Omega_{2,4,6,9,10} = 1/12$	$\Omega_{2,4,6,11,12} = 1/12$
$\Omega_{2,4,6,13,14} = 1/12$	$\Omega_{2,4,8,11,14} = -\sqrt{3}/36$	$\Omega_{2,4,8,12,13} = \sqrt{3}/36$
$\Omega_{2,4,11,14,15} = \sqrt{6}/36$	$\Omega_{2,4,12,13,15} = -\sqrt{6}/36$	$\Omega_{2,5,7,9,10} = 1/12$
$\Omega_{2,5,7,11,12} = 1/12$	$\Omega_{2,5,7,13,14} = 1/12$	$\Omega_{2,5,8,11,13} = -\sqrt{3}/36$
$\Omega_{2,5,8,12,14} = -\sqrt{3}/36$	$\Omega_{2,5,11,13,15} = \sqrt{6}/36$	$\Omega_{2,5,12,14,15} = \sqrt{6}/36$
$\Omega_{2,6,7,9,11} = 1/12$	$\Omega_{2,6,7,10,12} = 1/12$	$\Omega_{2,6,8,9,14} = \sqrt{3}/36$
$\Omega_{2,6,8,10,13} = -\sqrt{3}/36$	$\Omega_{2,6,9,14,15} = -\sqrt{6}/36$	$\Omega_{2,6,10,13,15} = \sqrt{6}/36$
$\Omega_{2,7,8,9,13} = \sqrt{3}/36$	$\Omega_{2,7,8,10,14} = \sqrt{3}/36$	$\Omega_{2,7,9,13,15} = -\sqrt{6}/36$
$\Omega_{2,7,10,14,15} = -\sqrt{6}/36$	$\Omega_{2,9,11,13,14} = -1/12$	$\Omega_{2,10,12,13,14} = -1/12$
$\Omega_{3,4,5,9,10} = 1/6$	$\Omega_{3,4,5,13,14} = 1/12$	$\Omega_{3,4,8,9,13} = -\sqrt{3}/36$
$\Omega_{3,4,8,10,14} = -\sqrt{3}/36$	$\Omega_{3,4,9,13,15} = \sqrt{6}/36$	$\Omega_{3,4,10,14,15} = \sqrt{6}/36$
$\Omega_{3,5,8,9,14} = \sqrt{3}/36$	$\Omega_{3,5,8,10,13} = -\sqrt{3}/36$	$\Omega_{3,5,9,14,15} = -\sqrt{6}/36$
$\Omega_{3,5,10,13,15} = \sqrt{6}/36$	$\Omega_{3,6,7,11,12} = -1/6$	$\Omega_{3,6,7,13,14} = -1/12$
$\Omega_{3,6,8,11,13} = \sqrt{3}/36$	$\Omega_{3,6,8,12,14} = \sqrt{3}/36$	$\Omega_{3,6,11,13,15} = -\sqrt{6}/36$
$\Omega_{3,6,12,14,15} = -\sqrt{6}/36$	$\Omega_{3,7,8,11,14} = -\sqrt{3}/36$	$\Omega_{3,7,8,12,13} = \sqrt{3}/36$

$\Omega_{3,7,11,14,15} = \sqrt{6}/36$	$\Omega_{3,7,12,13,15} = -\sqrt{6}/36$	$\Omega_{3,9,10,13,14} = -1/12$
$\Omega_{3,11,12,13,14} = 1/12$	$\Omega_{4,5,6,7,8} = -\sqrt{3}/6$	$\Omega_{4,5,6,11,14} = -1/12$
$\Omega_{4,5,6,12,13} = 1/12$	$\Omega_{4,5,7,11,13} = -1/12$	$\Omega_{4,5,7,12,14} = -1/12$
$\Omega_{4,5,8,9,10} = \sqrt{3}/9$	$\Omega_{4,5,8,13,14} = 5\sqrt{3}/36$	$\Omega_{4,5,9,10,15} = \sqrt{6}/18$
$\Omega_{4,5,13,14,15} = -\sqrt{6}/18$	$\Omega_{4,6,7,9,14} = -1/12$	$\Omega_{4,6,7,10,13} = 1/12$
$\Omega_{4,6,8,9,11} = \sqrt{3}/18$	$\Omega_{4,6,8,10,12} = \sqrt{3}/18$	$\Omega_{4,6,9,11,15} = \sqrt{6}/36$
$\Omega_{4,6,10,12,15} = \sqrt{6}/36$	$\Omega_{4,7,8,9,12} = \sqrt{3}/18$	$\Omega_{4,7,8,10,11} = -\sqrt{3}/18$
$\Omega_{4,7,9,12,15} = \sqrt{6}/36$	$\Omega_{4,7,10,11,15} = -\sqrt{6}/36$	$\Omega_{4,8,9,13,15} = -\sqrt{2}/12$
$\Omega_{4,8,10,14,15} = -\sqrt{2}/12$	$\Omega_{4,9,11,12,14} = -1/12$	$\Omega_{4,10,11,12,13} = 1/12$
$\Omega_{5,6,7,9,13} = -1/12$	$\Omega_{5,6,7,10,14} = -1/12$	$\Omega_{5,6,8,9,12} = -\sqrt{3}/18$
$\Omega_{5,6,8,10,11} = \sqrt{3}/18$	$\Omega_{5,6,9,12,15} = -\sqrt{6}/36$	$\Omega_{5,6,10,11,15} = \sqrt{6}/36$
$\Omega_{5,7,8,9,11} = \sqrt{3}/18$	$\Omega_{5,7,8,10,12} = \sqrt{3}/18$	$\Omega_{5,7,9,11,15} = \sqrt{6}/36$
$\Omega_{5,7,10,12,15} = \sqrt{6}/36$	$\Omega_{5,8,9,14,15} = \sqrt{2}/12$	$\Omega_{5,8,10,13,15} = -\sqrt{2}/12$
$\Omega_{5,9,11,12,13} = -1/12$	$\Omega_{5,10,11,12,14} = -1/12$	$\Omega_{6,7,8,11,12} = \sqrt{3}/9$
$\Omega_{6,7,8,13,14} = 5\sqrt{3}/36$	$\Omega_{6,7,11,12,15} = \sqrt{6}/18$	$\Omega_{6,7,13,14,15} = -\sqrt{6}/18$
$\Omega_{6,8,11,13,15} = -\sqrt{2}/12$	$\Omega_{6,8,12,14,15} = -\sqrt{2}/12$	$\Omega_{6,9,10,11,14} = -1/12$
$\Omega_{6,9,10,12,13} = 1/12$	$\Omega_{7,8,11,14,15} = \sqrt{2}/12$	$\Omega_{7,8,12,13,15} = -\sqrt{2}/12$
$\Omega_{7,9,10,11,13} = -1/12$	$\Omega_{7,9,10,12,14} = -1/12$	$\Omega_{8,9,10,11,12} = -\sqrt{3}/18$
$\Omega_{8,9,10,13,14} = \sqrt{3}/36$	$\Omega_{8,11,12,13,14} = \sqrt{3}/36$	$\Omega_{9,10,11,12,15} = -\sqrt{6}/9$
$\Omega_{9,10,13,14,15} = -\sqrt{6}/9$	$\Omega_{11,12,13,14,15} = -\sqrt{6}/9$	

Any pair of sets of non-zero coordinates (i_1, i_2, i_3) and $(i_4, i_5, i_6, i_7, i_8)$ for the $su(4)$ three- and five-cocycles with distinct index sets defines a non-zero coordinate of the seven-cocycle, the indices $(i_9, \ldots, i_{15})$ of which take the remaining available values in the set i=$(1, \cdots, 15)$ [see (8.15) below]. There are more than 400 non-zero such coordinates, which are not given here.

5 Primitive invariant symmetric polynomials, Casimirs and cocycles

5.1 General considerations and the case of $su(n)$

The polynomial ring of commuting operators in the enveloping algebra $\mathcal{U}(\mathcal{G})$ of a simple algebra $\mathcal{G}$ is freely generated by l Casimir-Racah operators [2, 3, 6, 4, 5, 7, 8, 9] of orders $m_1, \ldots, m_l$. As a result, the $k^{(m)}$ polynomials for $su(n)$, say, will be expressible in terms of the lower order primitive ones if $m > n = l{+}1$. For instance, $k_{i_1 i_2 i_3 i_4} = \frac{1}{4!}\mathrm{sTr}(X_{i_1} X_{i_2} X_{i_3} X_{i_4})$ for $su(4)$ is primitive and generates a (non-trivial) fourth-order Casimir, but it turns out to be proportional to $\delta_{(i_1 i_2}\delta_{i_3 i_4)}$ (and does not lead to a fourth-order primitive Casimir) when the $T_i \in su(3)$ (see Example 5.1 below and (6.9)). The case of $su(l+1)$ has been

considered before (see, *e.g.*, [31, 26, 25]) but to our knowledge there is no unified treatment available in the literature for the four simple infinite series.

Let us start by recalling the case of $su(n)$ [31] and consider

$$\epsilon^{\beta_1\ldots\beta_{n+1}}_{\alpha_1\ldots\alpha_{n+1}}(T_{i_1})^{\alpha_1}_{\cdot\beta_1}\cdots(T_{i_{n+1}})^{\alpha_{n+1}}_{\cdot\beta_{n+1}}=0 \quad . \tag{5.1}$$

This expression is symmetric in the generator indices $i_1\ldots i_{n+1}$ and is obviously zero since the matrix representation indices α,β range from 1 to n. Using (3.5), it follows that the above expression is a sum of $(n{+}1)!$-terms which may be grouped in classes, each class being a sum of terms all involving a product of the same number $\nu_1,\ldots,\nu_{n+1}$ of products of traces of products of $1,\ldots,n{+}1$ matrices respectively, where $1{\cdot}\nu_1{+}2{\cdot}\nu_2{+}\ldots{+}(n{+}1)\nu_{n+1}=(n{+}1)$. In other words, the different *types* of products appearing in (3.5) are characterised by the partitions of $(n+1)$ elements *i.e.* by the Young patterns (see, *e.g.*, [32]) associated with S_{n+1},

$$[(n+1)^{\nu_{n+1}},n^{\nu_n},\ldots,2^{\nu_2},1^{\nu_1}] \tag{5.2}$$

(obviously, $\nu_{n+1}=1$ or 0). All the elements of S_{n+1} for a given pattern (a set of fixed integers $\nu_1,\ldots,\nu_{n+1}$) determine products of traces of matrices with the same grouping pattern and, moreover, appear in (5.1) with the same sign (they correspond in S_{n+1} to the same conjugation class). The number of S_N elements associated with a given Young pattern is given by the 1844 Cauchy formula

$$\frac{N!}{\nu_1!2^{\nu_2}\nu_2!3^{\nu_3}\nu_3!\ldots N^{\nu_N}} \tag{5.3}$$

and since they correspond to permutations $\sigma\in S_N$ with (equal) parity $\pi(\sigma)$,

$$\pi(\sigma)=(-1)^{\nu_2+\nu_4+\nu_6+\ldots} \quad , \tag{5.4}$$

they all contribute to (5.1) with the same sign.

The mechanism which expresses the higher-order symmetric invariant polynomials in terms of the primitive ones is now clear. A given Young pattern determines a specific product of invariant symmetric tensors with the sign (5.4) and weighted by the factor (5.3). Since one of the terms in the sum (5.1) corresponds to the partition $\nu_1=\ldots=\nu_{N-1}=0\,,\,\nu_N=1$, it follows that the invariant symmetric tensor of order $m=n+1=l+2$ will be expressed, through (5.1), in terms of the l tensors of order $2,3,\ldots,l+1$ and that only these are primitive for $n=l+1$.

We are thus lead to the following

Lemma 5.1 (*Invariant symmetric polynomials on* $su(n)$)
There are l invariant polynomials of order $2,3,\ldots,l+1$; the others are not primitive, and may be expressed in terms of products of them.

The simplest application is that $d_{i_1i_2i_3}=0$ for $su(2)$. For the next higher order we have the following

Example 5.1
Let $\mathcal{G} = su(3)$. Using the λ_i matrices, $(i = 1, \ldots, 8)$, we find, since they are tridimensional,

$$\begin{aligned}&\epsilon^{\beta_1\beta_2\beta_3\beta_4}_{\alpha_1\alpha_2\alpha_3\alpha_4}(\lambda_{i_1})^{\alpha_1}_{\cdot\beta_1}(\lambda_{i_2})^{\alpha_2}_{\cdot\beta_2}(\lambda_{i_3})^{\alpha_3}_{\cdot\beta_3}(\lambda_{i_4})^{\alpha_4}_{\cdot\beta_4}\\ &= \mathrm{s}\{\frac{1}{2^2 2}\mathrm{Tr}(\lambda_{i_1}\lambda_{i_2})\mathrm{Tr}(\lambda_{i_3}\lambda_{i_4}) - \frac{1}{4}\mathrm{Tr}(\lambda_{i_1}\lambda_{i_2}\lambda_{i_3}\lambda_{i_4})\} = 0 \quad ,\end{aligned}$$

where, as before, s means symmetrisation in all indices i_1, i_2, i_3, i_4. Thus, the fourth-order symmetric tensor can be expressed in terms of the Killing second-order one. Using (4.2) we find

$$\mathrm{Tr}(\lambda_{(i_1}\lambda_{i_2}\lambda_{i_3}\lambda_{i_4)}) = 2\delta_{(i_1 i_2}\delta_{i_3 i_4)} \quad , \tag{5.5}$$

But using again (4.2) we may compute directly for $su(n)$

$$\frac{1}{4!}\mathrm{sTr}(\lambda_{i_1}\lambda_{i_2}\lambda_{i_3}\lambda_{i_4}) = \frac{1}{4\cdot 4!}\mathrm{sTr}(\{\lambda_{i_1},\lambda_{i_2}\}\{\lambda_{i_3},\lambda_{i_4}\}) = \frac{4}{n}\delta_{(i_1 i_2}\delta_{i_3 i_4)} + 2d_{\rho(i_1 i_2}d_{i_3 i_4)\rho} \ . \tag{5.6}$$

Eq. (5.5) and eq. (5.6) for $n = 3$ now give $d_{\rho(i_1 i_2}d_{i_3 i_4)\rho} = \frac{1}{3}\delta_{(i_1 i_2}\delta_{i_3 i_4)}$. For $su(3)$, the tensor $k_{i_1 i_2 i_3 i_4}$ [eq. (2.3)] is given by

$$k_{i_1 i_2 i_3 i_4} = \left(\frac{-i}{2}\right)^4 \mathrm{Tr}(\lambda_{(i_1}\lambda_{i_2}\lambda_{i_3}\lambda_{i_4)}) = \frac{1}{12}\delta_{(i_1 i_2}\delta_{i_3 i_4)} + \frac{1}{8}d_{\rho(i_1 i_2}d_{i_3 i_4)\rho} \quad . \tag{5.7}$$

5.2 The case of $so(n)$ and $sp(l)$

Let us now turn to the case of the orthogonal (odd, even) (B_l, D_l) and symplectic (C_l) algebras. In the defining representation these groups preserve the $n \times n$ euclidean or symplectic metric η and thus the generators X_i of these algebras satisfy

$$X_i\eta = -\eta X_i^t \quad , \tag{5.8}$$

where $i = 1, \ldots, l(2l+1)$, $l \geq 2$, $n = 2l+1$ (B_l); $i = 1, \ldots, l(2l+1)$, $l \geq 3$, $n = 2l$ (C_l); and $i = 1, \ldots, l(2l-1)$, $l \geq 4$, $n = 2l$ (D_l).

The symmetrised products of an *odd* number of generators of the orthogonal and symplectic groups also satisfies (5.8); hence, they are a member of their respective algebras. In particular, we may write

$$\{X_{i_1}, X_{i_2}, X_{i_3}\} = v_{i_1 i_2 i_3 .}{}^{\sigma} X_\sigma \quad , \tag{5.9}$$

where the bracket $\{\ ,\ldots,\ \}$ denotes symmetrisation *i.e.*, it is the sum of the 6 possible products and $v_{i_1 i_2 i_3 \sigma}$ is an invariant symmetric polynomial [18]. Now we may define a new $v^{(m)}$ family of invariant symmetric polynomials (cf. $d^{(m)}$ used in (6.1) for $su(n)$) by

$$v_{i_1 \ldots i_{2p}} = v_{(i_1 i_2 i_3 .}{}^{\alpha_1} v_{\alpha_1 i_4 i_5 .}{}^{\alpha_2} \cdots v_{\alpha_{p-2} i_{2p-2} i_{2p-1} i_{2p})} \tag{5.10}$$

With this notation (see (2.4))

$$
\begin{aligned}
k_{i_1\dots i_{2p}} &= \frac{1}{(2p)!}\mathrm{sTr}(X_{i_1}\cdots X_{i_{2p}}) \equiv \mathrm{Tr}(X_{(i_1}\cdots X_{i_{2p})}) \\
&= \frac{1}{6^{p-1}}\mathrm{Tr}(\{\dots\{\{X_{(i_1}, X_{i_2}, X_{i_3}\}, X_{i_4}, X_{i_5}\}, \dots, X_{i_{2p-2}}, X_{i_{2p-1}}\}X_{i_{2p})}) \\
&= \frac{1}{6^{p-1}} v_{(i_1 i_2 i_3 .}{}^{\alpha_1} v_{\alpha_1 i_4 i_5 .}{}^{\alpha_2} \cdots v_{\alpha_{p-2} i_{2p-2} i_{2p-1} .}{}^{\sigma}\mathrm{Tr}(X_\sigma X_{i_{2p})}) \\
&= \frac{\kappa}{6^{p-1}} v_{(i_1 i_2 i_3 .}{}^{\alpha_1} v_{\alpha_1 i_4 i_5 .}{}^{\alpha_2} \cdots v_{\alpha_{p-2} i_{2p-2} i_{2p-1} i_{2p})} \quad .
\end{aligned}
\tag{5.11}
$$

Thus,

$$
k_{i_1\dots i_{2p}} = \frac{\kappa}{6^{p-1}} v_{i_1\dots i_{2p}} \quad . \tag{5.12}
$$

This leads us to the following simple lemma

Lemma 5.2 (*Generation of higher order invariant symmetric polynomials for* B_l, C_l, D_l)
Let $v_{i_1 i_2 i_3 i_4}$ be the second lowest symmetric invariant polynomial for B_l, C_l, D_l. Then, the higher order symmetric polynomials $k_{i_1\dots i_{2p}}$ may be written in terms of $v_{i_1 i_2 i_3 i_4}$ by means of (5.12) and (5.10).

Let now $\mathcal{G}$ be B_l (C_l) and let X_i be a basis for B_l (C_l) given by $(2l{+}1)$- $(2l)$- dimensional matrices. Then, since the symmetric trace of a product of an odd number of X's is zero, and any partition of an odd number of elements will always include a symmetric trace of an odd number of X's we have to consider an even product

$$
\epsilon^{\beta_1\dots\beta_{2l+2}}_{\alpha_1\dots\alpha_{2l+2}} (X_{i_1})^{\alpha_1}_{.\beta_1} (X_{i_2})^{\alpha_2}_{.\beta_2} \cdots (X_{i_{2l+2}})^{\alpha_{2l+2}}_{.\beta_{2l+2}} = 0 \quad ; \tag{5.13}
$$

otherwise each term in the *l.h.s.* would be zero. Reasoning as before, but taking now into account that only traces of an *even* number of factors are different from zero, we are led to the following

Lemma 5.3 (*Invariant symmetric polynomials for* B_l, C_l)
The symmetric invariant polynomials for B_l, C_l given by (2.3) are all of even order $m = 2k$ and non-primitive for $m > 2l$. The relation which expresses the lowest non-primitive symmetric polynomial in terms of the primitive ones follows from the equality

$$
\begin{aligned}
\sum_{\text{partitions}} & \frac{(-1)^{\nu_2+\nu_4+\dots+\nu_{2l+2}}}{2^{\nu_2}\nu_2! 4^{\nu_4}\nu_4! \dots (2l+2)^{\nu_{2l+2}}} \\
& \cdot \mathrm{s}\left\{[\mathrm{Tr}(X_{i_1}X_{i_2})]^{\nu_2} [\mathrm{Tr}(X_{i_{2\nu_2+1}} X_{i_{2\nu_2+2}} X_{i_{2\nu_2+3}} X_{i_{2\nu_2+4}})]^{\nu_4} \dots\right\} = 0 \quad ,
\end{aligned}
\tag{5.14}
$$

where $\sum$ is extended over all partitions of $(2l+2)$ in (even) factors (all Young patterns of S_{2l+2}) and there is symmetrisation over all indices $i_1, \dots, i_{2l+2}$.

Example 5.2 For B_2 we obtain that

$$
\begin{aligned}
\epsilon^{\beta_1\ldots\beta_6}_{\alpha_1\ldots\alpha_6}(X_{i_1})^{\alpha_1}_{\cdot\beta_1}\ldots(X_{i_6})^{\alpha_6}_{\cdot\beta_6} &= \mathrm{s}\{-\frac{1}{3!2^3}\mathrm{Tr}(X_{i_1}X_{i_2})\mathrm{Tr}(X_{i_3}X_{i_4})\mathrm{Tr}(X_{i_5}X_{i_6}) \\
&+ \frac{1}{4\cdot 2}\mathrm{Tr}(X_{i_1}X_{i_2}X_{i_3}X_{i_4})\mathrm{Tr}(X_{i_5}X_{i_6}) - \frac{1}{6}\mathrm{Tr}(X_{i_1}\ldots X_{i_6})\} \\
&= 0 \quad ,
\end{aligned}
$$

relation which expresses the invariant polynomial of order 6, $\mathrm{Tr}(X_{i_1}\ldots X_{i_6})$, in terms of those of order 2 and 4.

Example 5.3 For C_3 we have

$$
\begin{aligned}
&\epsilon^{\beta_1\ldots\beta_8}_{\alpha_1\ldots\alpha_8}(X_{i_1})^{\alpha_1}_{\cdot\beta_1}\ldots(X_{i_8})^{\alpha_8}_{\cdot\beta_8} = \mathrm{s}\left\{\frac{1}{4!2^4}\mathrm{Tr}(X_{i_1}X_{i_2})\mathrm{Tr}(X_{i_3}X_{i_4})\mathrm{Tr}(X_{i_5}X_{i_6})\mathrm{Tr}(X_{i_7}X_{i_8})\right. \\
&-\frac{1}{2!2^24}\mathrm{Tr}(X_{i_1}X_{i_2})\mathrm{Tr}(X_{i_3}X_{i_4})\mathrm{Tr}(X_{i_5}X_{i_6}X_{i_7}X_{i_8}) + \frac{1}{12}\mathrm{Tr}(X_{i_1}X_{i_2})\mathrm{Tr}(X_{i_3}X_{i_4}X_{i_5}X_{i_6}X_{i_7}X_{i_8}) \\
&\left.+\frac{1}{2!4^2}\mathrm{Tr}(X_{i_1}X_{i_2}X_{i_3}X_{i_4})\mathrm{Tr}(X_{i_5}X_{i_6}X_{i_7}X_{i_8}) - \frac{1}{8}\mathrm{Tr}(X_{i_1}X_{i_2}X_{i_3}X_{i_4}X_{i_5}X_{i_6}X_{i_7}X_{i_8})\right\} = 0 \quad .
\end{aligned}
$$

This expression relates the eighth-order invariant symmetric polynomial with those of lower degree.

The case of the even orthogonal D_l is different because, in this case, there is an invariant polynomial which is related to the square root of the determinant of an even dimensional matrix (the Pfaffian).

Lemma 5.4 (*Invariant symmetric polynomials for D_l*)
The symmetric invariant polynomials for D_l given by (2.3) are of even order and primitive for $m = 2, 4 \ldots, 2l-2$. The higher order polynomials are written in terms of those and the polynomial $\mathrm{Pf}_{i_1\ldots i_l}$ constructed from the Pfaffian using that

$$
\epsilon^{\beta_1\ldots\beta_{2l}}_{\alpha_1\ldots\alpha_{2l}}(X_{i_1})^{\alpha_1}_{\cdot\beta_1}\ldots(X_{i_{2l}})^{\alpha_{2l}}_{\cdot\beta_{2l}} = \mathrm{Pf}_{(i_1\ldots i_l}\mathrm{Pf}_{i_{l+1}\ldots i_{2l})} \tag{5.15}
$$

which follows from the fact that $\det(\lambda^i X_i) = (\mathrm{Pf}(\lambda^i X_i))^2$ for the $\begin{pmatrix} 2l \\ l \end{pmatrix}$ skewsymmetric matrices X_i, and that these matrices constitute a basis in the vector space of $2l \times 2l$ skewsymmetric matrices.

Example 5.4 For D_4 we have

$$\epsilon^{\beta_1\ldots\beta_8}_{\alpha_1\ldots\alpha_8}(X_{i_1})^{\alpha_1}_{\cdot\beta_1}\ldots(X_{i_8})^{\alpha_8}_{\cdot\beta_8} = \mathrm{s}\left\{\frac{1}{4!2^4}\mathrm{Tr}(X_{i_1}X_{i_2})\mathrm{Tr}(X_{i_3}X_{i_4})\mathrm{Tr}(X_{i_5}X_{i_6})\mathrm{Tr}(X_{i_7}X_{i_8})\right.$$
$$-\frac{1}{2!2^2 4}\mathrm{Tr}(X_{i_1}X_{i_2})\mathrm{Tr}(X_{i_3}X_{i_4})\mathrm{Tr}(X_{i_5}X_{i_6}X_{i_7}X_{i_8}) + \frac{1}{12}\mathrm{Tr}(X_{i_1}X_{i_2})\mathrm{Tr}(X_{i_3}X_{i_4}X_{i_5}X_{i_6}X_{i_7}X_{i_8})$$
$$\left.+\frac{1}{2!4^2}\mathrm{Tr}(X_{i_1}X_{i_2}X_{i_3}X_{i_4})\mathrm{Tr}(X_{i_5}X_{i_6}X_{i_7}X_{i_8}) - \frac{1}{8}\mathrm{Tr}(X_{i_1}X_{i_2}X_{i_3}X_{i_4}X_{i_5}X_{i_6}X_{i_7}X_{i_8})\right\}$$
$$= \mathrm{Pf}_{(i_1\ldots i_4}\mathrm{Pf}_{i_5\ldots i_8)}$$

6 Invariant symmetric tensors for $su(n)$: a detailed study

The case of $su(n)$ is specially important since $SU(n)$-invariant tensors appear in many physical theories as *e.g.*, QCD. The properties of the $su(n)$-algebra tensors have already been discussed in [33, 29, 31, 26, 25] (and tables for the f_{ijk} and the d_{ijk} for $su(n)$ up to n=6 have been given in [30]), but we need to perform here a more complete and systematic study. The case of the cocycles was already discussed in Sec.4. We consider now the symmetric tensors, exhibiting in particular how the relations defining them for general n produce primitive tensors up to a given order m_l and non-primitive ones when this order is exceeded. To this aim, we shall use Lemma 3.3 and its Corollary 3.2.

6.1 d-tensors

We begin our study of totally symmetric tensors of arbitrary ranks for $su(n)$ by considering the d-tensor family [25] (see also [4]). For ranks $r = 2$ and 3 we have $d^{(2)}_{ij} = \delta_{ij}$, $d^{(3)}_{ijk} = d_{ijk}$, where the latter is the well-known totally symmetric and traceless $su(n)$ tensor, which exists for $n \geq 3$. Higher order tensors are defined recursively via

$$\begin{gathered} d^{(r+1)}_{i_1\ldots i_{r-1}i_r i_{r+1}} = d^{(r)}_{i_1\ldots i_{r-1}}{}^{j}\, d^{(3)}_{j i_r i_{r+1}} \quad , \quad r = 3, 4, \ldots \quad , \\ d^{(m)}_{i_1\ldots i_m} = d^{l_1}_{i_1 i_2} d^{l_2}_{l_1 i_3}\ldots d_{l_{m-3} i_{m-1} i_m} \end{gathered} \tag{6.1}$$

$i = 1, ..., (n^2-1)$. For $r \geq 3$, the above tensors are not symmetric in all their indices, so the required totally symmetric tensors are defined from their symmetrisation which gives

$$d^{(r)}_{(i_1\ldots i_r)} \quad . \tag{6.2}$$

Due to the fact that d_{ijk} is already symmetric, (6.2) normally represents the sum of $p < r!$ distinct terms divided by p. The construction defines a family of symmetric invariant tensors of order r.

We want to know the dimension and a basis for the vector space $\mathcal{V}^{(m)}$ of invariant symmetric tensors of a given order m (Corollary 3.2). For example, for n large enough

$\dim\mathcal{V}^{(4)} = 2$ and a basis is provided by

$$d^{(4)}_{(i_1 i_2 i_3 i_4)} = d^{(4)}_{(i_1 i_2 i_3) i_4} \quad , \quad \delta_{(i_1 i_2}\delta_{i_3 i_4)} = \delta_{(i_1 i_2}\delta_{i_3) i_4} \quad . \tag{6.3}$$

Similarly, also for n large enough $\dim\mathcal{V}^{(5)} = 2$ and a basis is provided by

$$d^{(5)}_{(i_1 i_2 i_3 i_4 i_5)} \quad , \quad d_{(i_1 i_2 i_3}\delta_{i_4 i_5)} \quad . \tag{6.4}$$

In fact, $\dim\mathcal{V}^{(4)} = 2$, only for $n \geq 4$, since if (and only if) $n = 3$, we have from Ex.5.1

$$d^{(4)}_{(i_1 i_2 i_3 i_4)} = \frac{1}{3}\delta_{(i_1 i_2}\delta_{i_3 i_4)} \quad . \tag{6.5}$$

Similarly, $\dim\mathcal{V}^{(5)} = 2$ only for $n \geq 5$, a basis being provided by $d^{(5)}_{(i_1 i_2 i_3 i_4 i_5)}$ and $d_{(i_1 i_2 i_3}\delta_{i_4 i_5)}$. For $n = 4$, however,

$$d^{(5)}_{(i_1 i_2 i_3 i_4 i_5)} = \frac{2}{3} d_{(i_1 i_2 i_3}\delta_{i_4 i_5)} \quad , \tag{6.6}$$

while, for $n = 3$, we have as a direct consequence of (6.5)

$$d^{(5)}_{(i_1 i_2 i_3 i_4 i_5)} = \frac{1}{3} d_{(i_1 i_2 i_3}\delta_{i_4 i_5)} \quad . \tag{6.7}$$

These results will serve as a check on some calculations done later. They are a consequence of Corollary 3.2; eqs. (6.5) and (6.6) also follow from work in [25] and in [31], both of which supply values for $\dim\mathcal{V}^{(4)}$ and suggest possible bases to use for increasing r. The recursive procedure of [25] actually supplies more information, but the work in [31] is also useful, providing direct access to results for each fixed n.

One way to see that not all families of symmetric invariant tensors of a given order are equally useful arises when one uses them to construct Casimir operators by (2.6)

$$\mathcal{C}^{(r)} = d^{(r)}_{(i_1 \ldots i_r)} X^{i_1} \ldots X^{i_r} \tag{6.8}$$

where the X_i are generators of $su(n)$. In principle this yields an arbitrary number of Casimir operators, one of each order $r \geq 2$. But since $su(n)$ has $(n-1)$ primitive Casimir operators of order $2, 3, \ldots, n$, it must be possible to express those with order $r > n$ in terms of the primitive ones. It is well known how to do this for $su(3)$, where (6.5), (6.7) imply

$$\begin{gathered}\mathcal{C}^{(4)} = \frac{1}{3}\delta_{(i_1 i_2}\delta_{i_3 i_4)} X^{i_1} X^{i_2} X^{i_3} X^{i_4} = \frac{1}{3}(\mathcal{C}^{(2)})^2 + \frac{1}{9} f_{i_1 i_2 i_3} X^{i_1} X^{i_2} X^{i_3} = \frac{1}{3}(\mathcal{C}^{(2)})^2 + \frac{1}{6}\mathcal{C}^{(2)} \quad , \\ \mathcal{C}^{(5)} = \frac{1}{3}\mathcal{C}^{(2)}\mathcal{C}^{(3)} + \frac{1}{4}\mathcal{C}^{(3)} \quad .\end{gathered} \tag{6.9}$$

Similarly, for $su(4)$ eq. (6.6) gives

$$\mathcal{C}^{(5)}=\frac{2}{3}\mathcal{C}^{(2)}\mathcal{C}^{(3)}+\frac{2}{3}\mathcal{C}^{(3)} \quad . \tag{6.10}$$

It is not possible in practice to treat such matters explicitly or even systematically for arbitrarily large n, r. It is thus convenient to replace the family of symmetrised d-tensors by a family for which no similar difficulties arise. This is achieved by using the t-tensors introduced in Lemma 3.2 and taking advantage of their property (3.18).

6.2 t-tensors

Working within the t family, we can build from $t^{(m)}$ only one non-vanishing scalar quantity,

$$K^{(m)}(n)=t^{(m)\, i_1\ldots i_m}t^{(m)}_{i_1\ldots i_m} \quad ; \tag{6.11}$$

all other full contractions are zero by Lemma 3.3. Since the standard $su(n)$ Gell-Mann matrices λ_i allow easy computation of the d-tensors, and a well-developed technology to operate with them exists (which can be completed when necessary with additional relations, see Appendix), we give the expressions of our t-tensors in terms of the d-tensors in Sec. 6.1. From (3.8) we trivially obtain

$$t_{i_1 i_2}=n\delta_{i_1 i_2} \quad . \tag{6.12}$$

Similarly, for m=3,4 eqs. (3.15), (3.6) give

$$t_{i_1 i_2 i_3}=\frac{n^2}{3}k_{i_1 i_2 i_3}=\frac{in^2}{12}d_{ijk} \quad , \tag{6.13}$$

$$t_{i_1 i_2 i_3 i_4}=\frac{1}{120}[n(n^2+1)d^{(4)}_{(i_1 i_2 i_3 i_4)}-2(n^2-4)\delta_{(i_1 i_2}\delta_{i_3 i_4)}] \quad , \tag{6.14}$$

and also

$$t_{i_1 i_2 i_3 i_4 i_5}=\lambda(n)[n(n^2+5)d^{(5)}_{(i_1 i_2 i_3 i_4 i_5)}-2(3n^2-20)d_{(i_1 i_2 i_3}\delta_{i_4 i_5)}] \quad , \tag{6.15}$$

by identifying $t_{i_1 i_2 i_3 i_4 i_5}$ as the traceless linear combination of the appropriate $\dim\mathcal{V}^{(5)}=2$ tensors. The factor $\lambda(n)$ has not been determined explicitly: the evaluation of (6.14) is already time consuming. These results apply to generic n, *i.e.* for n such that $\dim\mathcal{V}^{(4)}=2=\dim\mathcal{V}^{(5)}$. It is a valuable check on the computations to insert $n=3$ into (6.14) and $n=4$ into (6.14) and (6.15), and employ (6.5), (6.6) and (6.7) to obtain

$$\text{for } n=3: \quad t_{i_1 i_2 i_3 i_4}=\frac{1}{4}[d^{(4)}_{(i_1 i_2 i_3 i_4)}-\frac{1}{3}\delta_{(i_1 i_2}\delta_{i_3 i_4)}]=0 \quad ; \tag{6.16}$$

$$\text{for } n=3: \quad t_{i_1 i_2 i_3 i_4 i_5}=42\lambda(3)[d^{(5)}_{(i_1 i_2 i_3 i_4 i_5)}-\frac{1}{3}d_{(i_1 i_2 i_3}\delta_{i_4 i_5)}]=0 \quad ; \tag{6.17}$$

$$\text{for } n=4: \quad t_{i_1 i_2 i_3 i_4 i_5}=84\lambda(4)[d^{(5)}_{(i_1 i_2 i_3 i_4 i_5)}-\frac{2}{3}d_{(i_1 i_2 i_3}\delta_{i_4 i_5)}]=0 \quad . \tag{6.18}$$

These results exhibit the crucial property of the t-tensors. They are in one-to-one correspondence with the Ω tensors and hence, as we discuss below, provide an ideal way (free from problems associated with d-tensors noted above) to discuss Casimir operators. For low n, for which only $(2m-1)$-cocycles with $2m-1 \leq 2n-1$ can be non-zero, the generic results for $t^{(m)}$-tensors (6.12)-(6.15) collapse according to (6.16)-(6.18), as they must when $m > n$, to give vanishing tensors.

An alternative way to see the same mechanism at work stems from calculation of the numbers $K^{(m)}(n)$ of (6.11). We have

$$K^{(m)} = 0 \quad m > n \quad , \tag{6.19}$$

so that $su(n)$ only defines $(n-1)$ non-zero scalars. By direct computation based on (6.12)-(6.15) (using $d_{ijk}d_{ijk} = (n^2-1)(n^2-4)/n$ [*cf.* eq. (A.2)] and similar relations, see the Appendix) we get for $m = 2, 3$ and 4, with some work in the last case

$$K^{(2)}(n) = n^2(n^2-1) \tag{6.20}$$

$$K^{(3)}(n) = -\frac{1}{144}n^3(n^2-1)(n^2-4) \tag{6.21}$$

$$K^{(4)}(n) = \left(\frac{1}{120}\right)^2 \frac{2}{3}n^2(n^2+1)(n^2-1)(n^2-4)(n^2-9) \tag{6.22}$$

and likewise from (6.15)

$$K^{(5)}(n) = [\lambda(n)]^2 \frac{n}{3}(n^2+5)\prod_{l=1}^{4}(n^2-l^2) \quad . \tag{6.23}$$

Hence, it follows that

$$\begin{array}{ccccccccc} K^{(3)}(2) & = & K^{(4)}(2) & = & K^{(5)}(2) & = & \ldots & = & 0 \\ & & K^{(4)}(3) & = & K^{(5)}(3) & = & \ldots & = & 0 \\ & & & & K^{(5)}(4) & = & \ldots & = & 0 \end{array} \tag{6.24}$$

It is now plausible to conclude that the pattern persists for all $su(n)$ algebras, and that with the required modifications it remains true for other classical families.

For low rank groups, for which cocycles above a certain order cannot appear, the corresponding possibly non-zero scalars $K^{(m)}$ that can be formed can be seen to vanish identically, as expected since the t-tensors used to define them also do so.

6.3 Relations for the Casimir operators $\mathcal{C}'$

Corollary 3.3 exhibits the one-to-one correspondence among the $t^{(m)}$ tensors and the Casimir invariants $\mathcal{C}'^{(m)}$. The properties of the t tensors in Sec. 6.2 show now that we

do not get primitive Casimirs of undesired order, since (6.16), (6.17), for instance, yield $\mathcal{C}'^{(4)} = 0 = \mathcal{C}'^{(5)}$. This is of course needed to make sense: there are no $\Omega^{(7)}$, $\Omega^{(9)}$ cocycles for $su(3)$ since there are only l primitive Casimirs and cocycles for $su(l+1)$. Similarly (6.18) shows that $\mathcal{C}'^{(5)} = 0$ for $su(4)$ since there is no $\Omega^{(9)}$ for this algebra.

Example 6.1 For $su(n)$ ($n > 2$), eq. (3.19) gives

$$\mathcal{C}'^{(3)} = [\Omega^{(5)}]^{i_1 i_2 i_3 i_4 i_5} X_{i_1} X_{i_2} X_{i_3} X_{i_4} X_{i_5} = \frac{n^2}{12} k^{abc} X_a X_b X_c = \frac{in^2}{48} d^{abc} X_a X_b X_c \quad , \tag{6.25}$$

as expected from (3.20) and (6.13). Similarly, if we compute the 4th-order Casimir from (3.19), we obtain

$$\begin{aligned} \mathcal{C}'^{(4)} &= [\Omega^{(7)}]^{i_1 i_2 i_3 i_4 i_5 i_6 i_7} X_{i_1} X_{i_2} X_{i_3} X_{i_4} X_{i_5} X_{i_6} X_{i_7} \\ &= \frac{1}{120 \cdot 8} (n(n^2+1)\mathcal{C}^{(4)} - 2(n^2-4)\delta^{(i_1 i_2}\delta^{i_3 i_4)} X_{i_1} X_{i_2} X_{i_3} X_{i_4}) \\ &= \frac{1}{120 \cdot 8} (n(n^2+1)\mathcal{C}^{(4)} - 2(n^2-4)([\mathcal{C}^{(2)}]^2 + \frac{n}{6}\mathcal{C}^{(2)})) \quad . \end{aligned} \tag{6.26}$$

If we set $n = 3$ and use (6.9) above we get

$$\mathcal{C}'^{(4)} = \frac{1}{12 \cdot 8} (3\mathcal{C}^{(4)} - [\mathcal{C}^{(2)}]^2 - \frac{1}{2}\mathcal{C}^{(2)}) = 0 \quad . \tag{6.27}$$

Thus, with the $\mathcal{C}'$ family, we do not obtain Casimir operators beyond the order of the higher invariant primitive polynomial of the algebra. Also we have $\mathcal{C}'^{(2)} = f^{abc} X_a X_b X_c = (1/2) f^{abc} f_{abd} X^d X_c = (n/2) X^d X_d$. Recall that the fourth-order Casimir $\mathcal{C}^{(4)}$ is defined by $\mathcal{C}^{(4)} = d^{\rho (i_1 i_2} d_\rho^{i_3) i_4} X_{i_1} X_{i_2} X_{i_3} X_{i_4}$ (see (6.8)).

We may conclude by stating that the l tensors t introduced by Lemma 3.2 are in a rather privileged position due to their full 'tracelessness', eq. (3.18). In particular, eq. (3.19) provides a definition of the primitive Casimirs $\mathcal{C}'$ which does not contain non-primitive terms and which hence gives zero, as it should, when their order goes beyond the maximum m_l order permitted.

6.4 Some technical remarks

The crucial results (6.14) and (6.15), which provide explicit expressions for the fourth and fifth-order invariant symmetric $su(n)$-tensors require, for their derivation, a series of expressions involving properties of the f and d tensors of $su(n)$ which at present are not available in the literature. The same applies to the calculations (6.20)-(6.23) for the canonical $su(n)$ scalars $K^{(n)}$, on which we have based some generalisations. In addition to early work [29] containing a modest amount of generic $su(n)$ material, we have used [33] which contains a wide class of identities for f and d tensors. To proceed further here, we make use of the fact that an arbitrary symmetric $SU(n)$-invariant tensor may be expanded in terms of a basis of $\mathcal{V}^{(m)}$ as explained in Subsec.6.1 (see also [25, 31] for low values of m). We relegate the detailed treatment to an appendix.

7 Recurrence relations for primitive cocycles

Consider first the case of $su(n)$. In the defining representation, the unit matrix and the T_i's span a basis for the space of hermitian $n \times n$ matrices. This means that, since the symmetrised product of hermitian matrices is also a hermitian matrix, we can write

$$\{T_{i_1}, \ldots, T_{i_m}\} \propto \tilde{k}^{\sigma}_{i_1 \ldots i_m} T_\sigma + \hat{k}_{i_1 \ldots i_m} I \tag{7.1}$$

where $\tilde{k}^{\sigma}_{i_1 \ldots i_m}$ and $\hat{k}_{i_1 \ldots i_m}$ are invariant symmetric polynomials (of order $(m+1)$ and m) that can be related to the k tensors defined by (2.3). However, to give recurrence relations we want to use here the invariant symmetric tensors d ([25]; see also [4]) defined by eq. (6.1)[6].

The $d^{(m)}$ polynomials are not traceless, and hence differ from those of (3.15). Let us use them to define cocycles. Since the structure constants C_{ij}^k themselves provide a three-cocycle, the five-cocycle may be rewritten in the form

$$\Omega^{(5)}_{i_1 i_2 i_3 i_4 i_5} = \frac{1}{5!} \epsilon^{j_1 j_2 j_3 j_4 j_5}_{i_1 i_2 i_3 i_4 i_5} \Omega^{(3)}_{j_1 j_2 .}{}^{l_1} C^{l_2}_{j_3 j_4} d_{l_1 l_2 j_5} \quad . \tag{7.2}$$

The next case may be treated similarly. The coordinates of $\Omega^{(7)}$ are given by

$$\Omega^{(7)}_{i_1 i_2 i_3 i_4 i_5 i_6 i_7} = \frac{1}{7!} \epsilon^{j_1 j_2 j_3 j_4 j_5 j_6 j_7}_{i_1 i_2 i_3 i_4 i_5 i_6 i_7} C^{l_1}_{j_1 j_2} C^{l_2}_{j_3 j_4} C^{l_3}_{j_5 j_6} d^{(4)}_{l_1 l_2 l_3 j_7} \tag{7.3}$$

Now, although

$$d^{(4)}_{l_1 l_2 l_3 j_7} = d_{l_1 l_2 .}{}^{s} d_{s l_3 j_7} \quad . \tag{7.4}$$

(see (6.1)) is not fully symmetric, the extra skewsymmetry in (7.3) (cf. (3.4) and (3.6)) permits us to use (7.4) in (7.3) without having to symmetrise. [7] This means that (7.3) may be rewritten as

$$\begin{aligned}
\Omega^{(7)}_{i_1 i_2 i_3 i_4 i_5 i_6 i_7} &= \frac{1}{7!} \epsilon^{j_1 j_2 j_3 j_4 j_5 j_6 j_7}_{i_1 i_2 i_3 i_4 i_5 i_6 i_7} C^{l_1}_{j_1 j_2} C^{l_2}_{j_3 j_4} d_{l_1 l_2 .}{}^{s} C^{l_3}_{j_5 j_6} d_{s l_3 j_7} \\
&= \frac{1}{3!} \frac{1}{7!} \epsilon^{j_1 k_2 k_3 k_4 j_5 j_6 j_7}_{i_1 i_2 i_3 i_4 i_5 i_6 i_7} \epsilon^{j_2 j_3 j_4}_{k_2 k_3 k_4} C^{l_1}_{j_1 j_2} C^{l_2}_{j_3 j_4} d_{l_1 l_2 .}{}^{s} C^{l_3}_{j_5 j_6} d_{s l_3 j_7} \\
&= \frac{1}{7!} \epsilon^{j_1 k_2 k_3 k_4 j_5 j_6 j_7}_{i_1 i_2 i_3 i_4 i_5 i_6 i_7} \Omega^{(5)}_{j_1 k_2 k_3 k_4 .}{}^{s} C^{l_3}_{j_5 j_6} d_{s l_3 j_7} \quad .
\end{aligned} \tag{7.5}$$

Extending the procedure to an arbitrary $\Omega^{(2m-1)}$ cocycle we now get the following

[6]Other expressions such as (2.3) could be used here, since the non-primitive parts in (2.3) do not contribute to the cocycle definition due to Corollary 3.1.

[7]Note that, in the above examples, it is not necessary to introduce a five [seven] index ϵ tensor in the *r.h.s.*; instead, skewsymmetry in $i_2 i_3 i_4$ [$i_1 \ldots i_6$] will suffice in (7.2) [(7.5)] because $d_{(ijk)} = d_{ijk}$ [$d^{(4)}_{(ijkl)} = d^{(4)}_{(ijk)l}$]. However, for the fifth-order polynomial $d^{(5)}_{ijklm}$ we cannot use such a simplification (although in the symmetrisation there are fewer than 5! different terms). In this case (and for higher order d-polynomials) we need the complete $(2m-1)$-th order ϵ tensor, eq. (3.4), to remove the non-symmetric parts in (6.1).

Lemma 7.1 (*Recurrence relation for $su(n)$ primitive cocycles*)
Given a $(2(m-1)-1)$-cocycle of $su(n)$, the coordinates of the next $(2m-1)$-cocycle are obtained from $\Omega^{(2(m-1)-1)}$ by the formula

$$\Omega^{(2m-1)}_{i_1\ldots i_{2m-1}} = \frac{1}{(2m-1)!}\epsilon^{j_1\ldots j_{2m-1}}_{i_1\ldots i_{2m-1}}\Omega^{(2[m-1]-1)}_{j_1\ldots j_{2m-4}}{}^{s}_{\ .}C^{l}_{j_{2m-3}j_{2m-2}}d_{slj_{2m-1}} \quad . \tag{7.6}$$

Clearly, other relations may be found using again (7.6) to express the lower order cocycle $\Omega^{(2[m-1]-1)}$ in terms of $\Omega^{(2[m-2]-1)}$, etc.

For the symplectic and orthogonal algebras B_l, C_l, D_l (setting aside for D_l the case of the polynomial of order $m_l = l$ related to the Pfaffian) the primitive symmetric invariant polynomials of order $2, 4, \ldots, 2l$ (B_l, C_l) and $2, 4\ldots, (2l-2)$ (D_l) may be constructed by means of (5.10), and they lead to primitive cocycles of orders $3, 7, \ldots, (4l-1)$ (B_l, C_l) and of order $3, 7, \ldots, (4l-5)$ (D_l). Thus, the first recurrence relation starts for $\Omega^{(11)}$, which is written as

$$\Omega^{(11)}_{i_1\ldots i_{11}} = \frac{1}{11!}\epsilon^{j_1\ldots j_{11}}_{i_1\ldots i_{11}}C^{l_1}_{j_1j_2}\ldots C^{l_5}_{j_9j_{10}}v_{l_1\ldots l_5j_{11}} \tag{7.7}$$

or, using (5.10)

$$\begin{aligned}
\Omega^{(11)}_{i_1\ldots i_{11}} &= \frac{1}{11!}\epsilon^{j_1\ldots j_{11}}_{i_1\ldots i_{11}}C^{l_1}_{j_1j_2}C^{l_2}_{j_3j_4}C^{l_3}_{j_5j_6}C^{l_4}_{j_7j_8}C^{l_5}_{j_9j_{10}}v_{l_1l_2l_3.}{}^{s}v_{sl_4l_5j_{11}} \\
&= \frac{1}{5!}\frac{1}{11!}\epsilon^{j_1k_2\ldots k_6j_7\ldots j_{11}}_{i_1\ldots\ldots\ldots\ldots i_{11}}\epsilon^{j_2\ldots j_6}_{k_2\ldots k_6}C^{l_1}_{j_1j_2}C^{l_2}_{j_3j_4}C^{l_3}_{j_5j_6}v_{l_1l_2l_3.}{}^{s}C^{l_4}_{j_7j_8}C^{l_5}_{j_9j_{10}}v_{sl_4l_5j_{11}} \\
&= \frac{1}{11!}\epsilon^{j_1\ldots j_{11}}_{i_1\ldots i_{11}}\Omega^{(7)}_{j_1\ldots j_6.}{}^{s}C^{l_4}_{j_7j_8}C^{l_5}_{j_9j_{10}}v_{sl_4l_5j_{11}} \quad ,
\end{aligned} \tag{7.8}$$

which expresses $\Omega^{(11)}$ in terms of $\Omega^{(7)}$ and the fourth-order polynomial. This leads to the following recurrence relation

Lemma 7.2 (*Recurrence relation for $so(2l+1), sp(l), so(2l)$*)
Let $\Omega^{(4p-1)}$ be a $(4p-1)$-cocycle for $so(2l+1)$, $sp(l)$ $(p = 1, \ldots, l)$, $so(2l)$ $(p = 1, \ldots, (l-1))$. Then,

$$\Omega^{(4p-1)}_{i_1\ldots i_{4p-1}} = \frac{1}{(4p-1)!}\epsilon^{j_1\ldots j_{4p-1}}_{i_1\ldots i_{4p-1}}\Omega^{(4[p-1]-1)}_{j_1\ldots j_{4p-6}}{}^{s}_{\ .}C^{l_1}_{j_{4p-5}j_{4p-4}}C^{l_2}_{j_{4p-3}j_{4p-2}}v_{sl_1l_2j_{4p-1}} \,. \tag{7.9}$$

8 Duality relations for skewsymmetric primitive tensors

The interpretation of the primitive cocycles as closed forms on the group manifold of a simple compact group G provides another intuitive way to obtain additional relations among them. Consider the case of $su(2)$. The identification of the $su(2)$-three-cocycle with the closed de Rham three-form on $SU(2) \sim S^3$ tells us that this form is (up to a constant) the volume form on the $SU(2)$ group manifold. It is known [10, 11, 12, 13, 1, 14, 15, 16, 17]

that, from the point of view of real homology, the compact groups behave as 'products' of spheres of odd dimension $(2m_i - 1)$, $i = 1, \ldots, l$. As a result,

$$\Omega = \Omega^{(2m_1-1)} \wedge \ldots \wedge \Omega^{(2m_l-1)} \tag{8.1}$$

is proportional to the volume form on the group manifold and, indeed, $\sum_{i=1}^{l}(2m_i - 1) = r = \mathrm{dim}G$ for all simple groups. For instance, if $G = SU(3)$

$$\Omega(g) = \Omega^{(3)}(g) \wedge \Omega^{(5)}(g) \propto C_{i_1 i_2 i_3} \Omega^{(5)}_{i_4 i_5 i_6 i_7 i_8} \omega^{i_1}(g) \wedge \ldots \wedge \omega^{i_8}(g) \tag{8.2}$$

is proportional to $\omega^1(g) \wedge \ldots \wedge \omega^8(g)$, the volume element on $SU(3)$.

Algebraically $(\omega^i(g) \to \omega^i)$, we may look at (8.2) as the wedge product of two skewsymmetric tensors defined on a vector space V of dimension r. If we now endow V with a metric (the unit metric for $\mathcal{G}$ compact) we may introduce the Hodge $*$ operator. Then, the scalar product of two skewsymmetric tensors $\alpha = \frac{1}{q!}\alpha_{i_1 \ldots i_q}\omega^{(i_1)} \wedge \ldots \wedge \omega^{(i_q)}$ and β of order q is given by

$$< \alpha, \beta >= \alpha \wedge (*\beta) = \frac{1}{q!}\alpha_{i_1 \ldots i_q}\beta^{i_1 \ldots i_q}\omega^1 \wedge \ldots \wedge \omega^r \quad . \tag{8.3}$$

Consider the simplest $su(2)$ case. There is only one cocycle ($cf.$ (3.3))

$$\Omega^{(3)} = \frac{1}{3!}\Omega_{i_1 i_2 i_3}\omega^{i_1} \wedge \omega^{i_2} \wedge \omega^{i_3} \quad . \tag{8.4}$$

We fix the normalisation by demanding that

$$< \Omega^{(3)}, \Omega^{(3)} >= \Omega^{(3)} \wedge (*\Omega^{(3)}) = \frac{1}{3!}\Omega_{i_1 i_2 i_3}\Omega^{i_1 i_2 i_3}\omega^1 \wedge \omega^2 \wedge \omega^3 = \omega^1 \wedge \omega^2 \wedge \omega^3 \tag{8.5}$$

$(\omega^1(g) \wedge \omega^2(g) \wedge \omega^3(g)$ is the volume element on $SU(2))$, *i.e.*,

$$\frac{1}{3!}\Omega_{i_1 i_2 i_3}\Omega^{i_1 i_2 i_3} = 1 \quad , \tag{8.6}$$

which is trivially satisfied since $\Omega_{i_1 i_2 i_3} = \epsilon_{i_1 i_2 i_3}$ for $su(2)$. Let now $G = SU(3)$, and let the five cocycle be expressed as by

$$\Omega^{(5)} = \frac{1}{5!}\Omega_{i_1 i_2 i_3 i_4 i_5}\omega^{i_1} \wedge \omega^{i_2} \wedge \omega^{i_3} \wedge \omega^{i_4} \wedge \omega^{i_5} \quad . \tag{8.7}$$

We now fix now the normalisations of $\Omega^{(3)}$ and $\Omega^{(5)}$ by requiring that

$$\Omega^{(3)} \wedge (*\Omega^{(3)}) = \omega^1 \wedge \ldots \wedge \omega^8 = \Omega^{(5)} \wedge (*\Omega^{(5)}) \tag{8.8}$$

This gives the previous relation for the coordinates of $\Omega^{(3)}$ and a similar one for those of $\Omega^{(5)}$. Up to an irrelevant sign (which is a minus sign for even r, as is the case of $SU(3)$, since $*^2 = (-1)^{q(r-q)}$ where q is the order (always odd) of the cocycle) we may write

$$\Omega^{(5)} = *\Omega^{(3)} \tag{8.9}$$

(for a positive definite metric $< \alpha, \beta >=< *\alpha, *\beta >$) and hence (with r=8)

$$\Omega^{(5)} = \frac{1}{(r-3)!}\frac{1}{3!}\delta^{i_1j_1}\delta^{i_2j_2}\delta^{i_3j_3}\epsilon_{j_1j_2j_3l_1l_2l_3l_4l_5}\Omega^{(3)}_{i_1i_2i_3}\omega^{l_1}\wedge\omega^{l_2}\wedge\omega^{l_3}\wedge\omega^{l_4}\wedge\omega^{l_5} \tag{8.10}$$

i.e.,

$$\Omega^{(5)}_{l_1l_2l_3l_4l_5} = \frac{1}{3!}\epsilon_{j_1j_2j_3l_1l_2l_3l_4l_5}\Omega^{(3)j_1j_2j_3} \quad . \tag{8.11}$$

The previous arguments are not restricted to $su(n)$ nor to the case of two cocycles. In general we have that relation (8.1) holds and, as a result, we find up to irrelevant signs a whole series of duality relations among cocycles:

$$\begin{aligned}
&\Omega^{(2m_i-1)} = *(\Omega^{(2m_1-1)}\wedge\ldots\widehat{\Omega^{(2m_i-1)}}\wedge\ldots\Omega^{(2m_l-1)}) \\
&\Omega^{(2m_i-1)}\wedge\Omega^{(2m_j-1)} = *(\Omega^{(2m_1-1)}\wedge\ldots\widehat{\Omega^{(2m_i-1)}}\wedge\ldots\widehat{\Omega^{(2m_j-1)}}\wedge\ldots\Omega^{(2m_l-1)}) \\
&\ldots\ldots\ldots
\end{aligned} \tag{8.12}$$

where $1\leq i,j,\ldots\leq l$, etc. In general, the normalisation of the $(2m_i-1)$ cocycles may be introduced by requiring that

$$< \Omega^{(2m_i-1)}, \Omega^{(2m_i-1)} >= \Omega^{(2m_i-1)}\wedge(*\Omega^{(2m_i-1)}) = \omega^1\wedge\ldots\wedge\omega^r \quad , \tag{8.13}$$

the volume element on G.

Example 8.1 The three- and five-cocycles for $su(3)$, given in co-ordinates in tables 4.1 and 4.3 respectively, satisfy the duality relation

$$\Omega_{i_1i_2i_3i_4i_5} = \frac{1}{3!2\sqrt{3}}\epsilon_{i_1i_2i_3i_4i_5i_6i_7i_8}f^{i_6i_7i_8} \tag{8.14}$$

Example 8.2 For $su(4)$, we use the definitions of the three- and five-cocycles of tables 4.4 and 4.6 and the expression (7.5) of the seven-cocycle $\Omega^{(7)}$ to obtain the following relationship:

$$15\sqrt{2}\Omega^{(7)}_{i_1i_2i_3i_4i_5i_6i_7} = \frac{1}{5!3!}\epsilon_{i_1i_2i_3i_4i_5i_6i_7j_1j_2j_3j_4j_5k_1k_2k_3}\Omega^{(5)}_{j_1j_2j_3j_4j_5}\Omega^{(3)}_{k_1k_2k_3} \tag{8.15}$$

These relations have been computed using MAPLE and provide a further check of the cocycle Tables in Sec. 4.

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Appendix A: Traces of products of $su(n)$ D and F matrices

We define the hermitian D and antihermitian F (adjoint) traceless matrices for arbitrary $su(n)$ by

$$(F_a)_{bc} = f_{bac} \quad , \quad (D_a)_{bc} = d_{abc} \quad , \quad a,b,c = 1,...,n^2-1 \quad , \tag{A.1}$$

intending to present the identities that we require involving d and f tensors of $SU(n)$ in terms of traces of products of D and F matrices. All the 2 and 3-fold traces have been known for a long time [33, 29]. Explicitly,

$$\begin{array}{lll} \mathrm{Tr}F_aF_b = -n\delta_{ab}, & \mathrm{Tr}F_aD_b = 0, & \mathrm{Tr}D_aD_b = \frac{n^2-4}{n}\delta_{ab}, \\ \mathrm{Tr}F_aF_bF_c = -\frac{n}{2}f_{abc}, & \mathrm{Tr}F_aF_bD_c = -\frac{n}{2}d_{abc}, & \\ \mathrm{Tr}F_aD_bD_c = \frac{n^2-4}{2n}f_{abc}, & \mathrm{Tr}D_aD_bD_c = \frac{n^2-12}{2n}d_{abc} \quad . & \end{array} \tag{A.2}$$

The methods of [33] yield only expressions for such four-fold traces as

$$\mathrm{Tr}F_aF_bF_cD_d \quad , \quad \mathrm{Tr}F_aD_bD_cD_d, \tag{A.3}$$

as well as all others that follow from these which involve an odd number of F and D matrices. To proceed further (to the evaluation of the traces of all four-fold products of even numbers of D and F matrices), we begin by treating

$$\mathrm{Tr}F_{(a}F_bF_cF_{d)} \quad \text{and} \quad \mathrm{Tr}D_{(a}D_bD_cD_{d)} \quad . \tag{A.4}$$

Once this is done, $\mathrm{Tr}F_aF_bF_cF_d$, $\mathrm{Tr}D_aD_bD_cD_d$, $\mathrm{Tr}F_aF_bD_cD_d$ (and other similar traces) can be calculated by means of further elementary procedures. We list results valid for arbitrary $su(n)$.

$$\mathrm{Tr}F_aF_bF_cF_d = \delta_{ab}\delta_{cd} + \delta_{ad}\delta_{bc} + \frac{n}{4}(d_{abx}d_{cdx} + d_{adx}d_{bcx} - d_{acx}d_{bdx}), \tag{A.5}$$

$$\mathrm{Tr}F_aF_bF_cD_d = -\frac{n}{4}d_{abx}f_{cdx} - \frac{n}{4}f_{abx}d_{cdx}, \tag{A.6}$$

$$\begin{aligned} \mathrm{Tr}F_aF_bD_cD_d = {} & \frac{4-n^2}{n^2}(\delta_{ab}\delta_{cd} - \delta_{ac}\delta_{bd}) + \frac{8-n^2}{4n}(d_{abx}d_{cdx} - d_{acx}d_{bdx}) \\ & - \frac{n}{4}d_{adx}d_{bcx}, \end{aligned} \tag{A.7}$$

$$\mathrm{Tr}F_aD_bF_cD_d = \frac{n}{4}(d_{acx}d_{bdx} - d_{adx}d_{bcx}) - \frac{n}{4}d_{abx}d_{cdx}, \tag{A.8}$$

$$\begin{aligned}
\mathrm{Tr} F_a D_b D_c D_d &= \frac{n^2-12}{4n} f_{abx} d_{cdx} + \frac{n}{4} d_{abx} f_{cdx} \\
&+ \frac{1}{n}(f_{adx} d_{bcx} - f_{acx} d_{bdx}), \qquad \text{(A.9)}
\end{aligned}$$

$$\begin{aligned}
\mathrm{Tr} D_a D_b D_c D_d &= \frac{n^2-4}{n^2}(\delta_{ab}\delta_{cd} + \delta_{ad}\delta_{bc}) - \frac{n}{4} d_{acx} d_{bdx} \\
&+ \frac{n^2-16}{4n}(d_{abx} d_{cdx} + d_{adx} d_{bcx}). \qquad \text{(A.10)}
\end{aligned}$$

Now we illustrate the method of derivation of the above results and perform a variety of checks of their correctness. In the framework of Sec. 6, we set out from an expansion of an arbitrary totally symmetric fourth-rank tensor in $\mathcal{V}^{(4)}$ and write

$$\mathrm{Tr} F_{(a} F_b F_c F_{d)} = A d^{(4)}_{(abcd)} + B \delta_{(ab}\delta_{cd)} \quad . \qquad \text{(A.11)}$$

Contracting both sides with δ_{ab} and d_{abe} in turn and using results such as (A.2) allows us easily to find $A = n/4$, $B = 2$. Various checks on, *e.g.*, (A.5) and its consequences now exist. Firstly, in [33] we find identities for contractions of (A.5), (A.7) and (A.10) with d_{ace} and f_{ace}. Performing such contractions explicitly on our expressions for these traces, we find total agreement for all $SU(n)$. For the case of $SU(3)$ a more elementary derivation of, *e.g.* $\mathrm{Tr} F_a F_b F_c F_d$ is available because additional identities for $SU(3)$ D and F matrices exist as described in [29]. One can thus get the $n = 3$ version of, *e.g.* $\mathrm{Tr} F_{(a} F_b F_c F_{d)}$ without using any assumptions about symmetric tensors. In particular, for $SU(3)$, we have

$$\mathrm{Tr} F_{(a} F_b F_c F_{d)} = \frac{9}{4}\delta_{(ab}\delta_{cd)}, \qquad \text{(A.12)}$$

$$\mathrm{Tr} D_{(a} D_b D_c D_{d)} = \frac{17}{36}\delta_{(ab}\delta_{cd)}, \qquad \text{(A.13)}$$

$$\mathrm{Tr} D_a D_b D_c D_d = \frac{5}{9}\left(\delta_{ab}\delta_{cd} + \delta_{ad}\delta_{bc}\right) - \frac{7}{12}\delta_{(ab}\delta_{cd)} - \frac{1}{6} d_{acx} d_{bdx}. \qquad \text{(A.14)}$$

Analogously, from

$$\mathrm{Tr} D_{(a} D_b D_c D_{d)} = A' d^{(4)}_{(abcd)} + B' \delta_{(ab}\delta_{cd)} \qquad \text{(A.15)}$$

and contracting with δ_{ab} and d_{abs} we deduce that $A' = (n^2-32)/4n$ and $B' = 2(n^2-4)/n^2$.

Armed with the results (A.5) to (A.10) as well as those of (A.2), we can consider five-fold traces. Thus we postulate (see (6.4))

$$\mathrm{Tr} D_{(a} D_b D_c D_d D_{e)} = A d^{(5)}_{(abcde)} + B \delta_{(ab} d_{cde)} \quad , \qquad \text{(A.16)}$$

Contracting this with δ_{ab} and d_{abf} in turn gives three linear equations for the coefficients A and B; in the latter construction equating coefficients of $d^{(4)}_{(cde)f} = d^{(4)}_{(cdef)}$ and $\delta_{(cd}\delta_{e)f} =$

$\delta_{(cd}\delta_{ef)}$ has actually given rise to two equations. These three equations are consistent and give

$$A = \frac{n^2 - 80}{8n} \quad , \quad B = \frac{3n^2 - 20}{n^2} \quad . \tag{A.17}$$

The major cancellations that bring (A.17) to the form displayed convince us of the correctness of our results. There are also other independent checks available because results for various five-fold traces with no more than three free indices (vertex corrections, in the diagrammatic language) are known, which we can reproduce. We also note that (A.16) simplifies, not only in the case of $SU(3)$ when (6.7) is used to obtain

$$\mathrm{Tr} D_{(a} D_b D_c D_d D_{e)} = -\frac{5}{24} d_{(abc}\delta_{de)} \quad , \tag{A.18}$$

but also for $SU(4)$, after use of (6.6), giving

$$\mathrm{Tr} D_{(a} D_b D_c D_d D_{e)} = \frac{5}{12} d_{(abc}\delta_{de)}. \tag{A.19}$$

Note that (A.18) and (A.19) may be obtained by contracting the expression

$$\mathrm{Tr} D_{(a} D_b D_c D_d D_{e)} = A'' d_{(abc}\delta_{de)} \tag{A.20}$$

with δ_{ab}; the resulting equation gives $A'' = -\frac{5}{24}$ in the $su(3)$ case and $A'' = \frac{5}{12}$ for $su(4)$.

Finally, we note that, once (A.16) has been evaluated and further five-fold traces obtained from it by elementary procedures, we have all we need to establish the result

$$\begin{aligned} \mathrm{Tr} D_{(a} D_b D_c D_d D_e D_{f)} \quad &= \quad 4\frac{n^2-4}{n^3}\delta_{(ab}\delta_{cd}\delta_{ef)} + \frac{n^2-192}{16n} d_{(ab}{}^x d_{cd}{}^y d_{ef)}{}^z d_{xyz} \\ &+ \quad \frac{3}{4}\frac{3n^2-64}{n^2}\delta_{(ab} d_{cd}{}^x d_{ef)x} + \frac{5n^2+48}{4n^2} d_{(abc} d_{def)}. \end{aligned} \tag{A.21}$$

This involves expansion of the left side in terms of a basis [25, 31], of totally symmetric sixth-rank tensors. Various contractions give four linear equations for the coefficients involved. Regarding the expansion (A.21) we remark that the tensor $d_{(ab}{}^j\, d^{jk}{}_c\, d^{kl}{}_d\, d^l{}_{ef)}$ differs from the tensor of the second term by $2/n$ times the difference of the tensors of the fourth minus the first term.

Some further comments are now in order. Many of the procedures followed above have been guided by graphical ideas such as described in [34]. These simplify complicated expressions involving d and f tensors by reference to closed loops in their graphical representations. These loops correspond to traces and we set about the simplification of three line loops with the aid of (A.2). Then we learn how to handle in turn all loops of four and five internal lines. The calculations associated with (A.21) are organised by looking at contractions diagrammatically, preferring to simplify at all times by identifying loops of as few lines as possible. It should be noted that many of the identities of this appendix have been evaluated by different means including the use of MAPLE. Also. we emphasise

that a large number of checks of our results were made by passing to subcases for which identities were given in [33].

We turn next to the use of the identities presented in this appendix in the derivation of results quoted in the body of the paper. Consider, *e.g.* (6.14)

$$t_{p_1p_2p_3p_4} = Ad^{(4)}_{(p_1p_2p_3p_4)} + B\delta_{(p_1p_2}\delta_{p_3p_4)}. \tag{A.22}$$

Contractions with $\delta_{p_1p_2}$ and $d_{p_1p_2q}$ on the right are easy to do, as is the contraction of the left side with $\delta_{p_1p_2}$. The latter gives zero by direct calculation as required by Lemma 3.2. The contraction $t_{p_1p_2p_3p_4}d_{p_1p_2q}$ is not governed by any general argument. To compute it, we set out from (see (3.15)),

$$t_{pqrg} = \Omega^{(7)}_{abcdefg}f_{abp}f_{cdq}f_{efr}, \tag{A.23}$$

and the cocycles defined using the d-family

$$\Omega^{(7)}_{abcdefg} = \Omega^{(5)}_{t[abcd}f^{z}{}_{ef]}d_{tzg}, \tag{A.24}$$

$$\Omega^{(5)}_{tabcd}f_{abs} = \frac{n}{2}f_{u[cd}d_{t]us}, \tag{A.25}$$

((7.5), *cf.* (7.2)) for $SU(n)$. Patient evaluation of the terms involved can be completed with the aid of four-fold traces evaluated earlier in this appendix and not (it seems) without them. It is clear that the occurrence of f-tensors in the cocycle definitions is what requires us to know how to treat traces involving F as well as D matrices. Equation (6.14) emerges after typical and reassuring cancellations. To obtain (6.15) with $\lambda(n)$ left undetermined, it is necessary only to compute the ratio of the two scalars occurring in the expansion with respect to the basis [25, 31] of $\mathcal{V}^{(5)}$ of the left side of (6.15). This requires only contraction with $\delta_{p_1p_2}$ which is zero.

To prove the result for $K^{(5)}(n)$ in (6.23), we set out from (6.15). The only hard part involves

$$d^{(5)}_{(abcde)}d^{(5)}_{(abcde)} = d^{(5)}_{(abcde)}d^{(5)}_{abcde} = d^{(5)}_{(abcde)}d_{abx}d_{xcy}d_{yde} \quad , \tag{A.26}$$

and at worst two equal terms of the fifteen involved lead to a contracted four-fold trace of D matrices known from [33]. The result is $d^{(5)}_{(abcde)}d^{(5)}_{abcde} = \frac{(n^2-4)(n^2-1)}{15n^3}(5n^4 - 96n^2 + 480)$.

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